

MATH 51 FINAL EXAM SOLUTIONS (AUTUMN 2000)

1. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

(a) (6 points) Find the dimension of $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

Solution. Let

$$A = \begin{bmatrix} 1 & 1 & 3 \\ -1 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix}$$

Then

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so $\dim(\text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)) = \dim(C(A)) = 2$.

(b) (8 points) Find all vectors \mathbf{v} which are simultaneously orthogonal (i.e. perpendicular) to all three vectors $\mathbf{u}_1, \mathbf{u}_2$ and \mathbf{u}_3 .

Solution. The set of all such vectors is the null space of

$$B = \begin{bmatrix} 1 & -1 & 3 \\ 1 & 1 & -2 \\ 3 & -1 & 4 \end{bmatrix}.$$

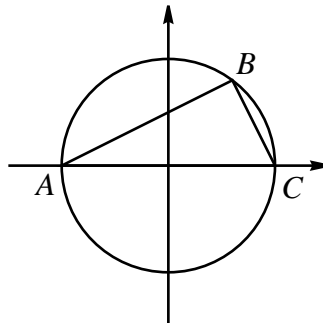
Since

$$\text{rref}(B) = \begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix},$$

it follows that

$$N(B) = \text{span} \left(\begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix} \right).$$

2. (10 points) Suppose $B = (x, y)$ is a point on the circle of radius 1 centered at the origin. That is, x and y satisfy $x^2 + y^2 = 1$. Let $A = (-1, 0)$, $C = (1, 0)$ and assume $y \neq 0$ (so that B is not equal to A or C).



Use dot products to show that angle ABC is a right angle.

Solution. The vector from B to A is

$$\mathbf{v}_1 = \begin{bmatrix} -1 - x \\ -y \end{bmatrix}$$

and the vector from B to C is

$$\mathbf{v}_2 = \begin{bmatrix} 1 - x \\ -y \end{bmatrix}.$$

Thus

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (-1 - x)(1 - x) + y^2 = -1 + x^2 + y^2 = 0$$

since $x^2 + y^2 = 1$. Therefore \mathbf{v}_1 and \mathbf{v}_2 are orthogonal. Since $y \neq 0$, these vectors are nonzero, so the angle between them is $\pi/2$.

3. Suppose A is a 5×5 matrix with

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For each part below, give the answer when possible. Otherwise answer “not enough information”.

(a) (2 points) Find a basis for $N(A)$.

Solution.

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

(b) (2 points) Find $\dim(N(A))$.

Solution. 2.

(c) (2 points) Find a basis for $C(A)$.

Solution. Not enough information.

(d) (2 points) Find $\dim(C(A))$.

Solution. 3.

(e) (2 points) Find the rank of A .

Solution. 3.

(f) (2 points) Find a vector $\mathbf{b} \in \mathbf{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions.

Solution. Not enough information.

(g) (2 points) Are there vectors $\mathbf{b} \in \mathbf{R}^5$ such that $A\mathbf{x} = \mathbf{b}$ has *exactly* one solution?

Solution. No. There are free variables.

(h) (2 points) Find the eigenvalues of A .

Solution. Not enough information.

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$$

(a) (5 points) Compute $\det(A)$.

Solution. $\det(A) = 1(3 - 0) - 1(6 - 4) + 0(0 - 2) = 1$.

(b) (7 points) Find A^{-1} .

Solution. Since

$$\text{rref} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 2 & 1 & 2 & 0 & 1 & 0 \\ 2 & 0 & 3 & 0 & 0 & 1 \end{array} \right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -3 & 2 \\ 0 & 1 & 0 & -2 & 3 & -2 \\ 0 & 0 & 1 & -2 & 2 & -1 \end{array} \right],$$

$$A^{-1} = \begin{bmatrix} 3 & -3 & 2 \\ -2 & 3 & -2 \\ -2 & 2 & -1 \end{bmatrix}.$$

5. (a) (6 points) Find the eigenvalues of A .

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 7 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution.

$$\det(\lambda I_3 - A) = \det \begin{bmatrix} \lambda - 2 & -4 & -6 \\ 0 & \lambda - 7 & -8 \\ 0 & 0 & \lambda - 3 \end{bmatrix} = (\lambda - 2)(\lambda - 7)(\lambda - 3),$$

so the eigenvalues of A are 2, 7 and 3.

(b) (8 points) Let

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$$

$\lambda = 3$ is an eigenvalue of B (you do not need to verify this). Find a basis for the eigenspace $E_3 = \{\mathbf{v} \in \mathbf{R}^3 \mid B\mathbf{v} = 3\mathbf{v}\}$.

Solution.

$$\text{rref}(3I_3 - B) = \text{rref} \begin{bmatrix} 1 & -2 & -1 \\ -1 & 2 & 1 \\ -2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

so

$$E_3 = N(3I_3 - B) = \text{span} \left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

6. (a) (5 points) Show that, for each choice of fixed vectors $\mathbf{b} \in \mathbf{R}^3$ and $\mathbf{c} \in \mathbf{R}^2$, the formula

$$T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{b})\mathbf{c}$$

defines a linear transformation $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$.

Solution. Using the properties of dot products and scalar multiplication, we have

$$\mathbf{T}(\mathbf{x} + \mathbf{y}) = ((\mathbf{x} + \mathbf{y}) \cdot \mathbf{b}) \cdot \mathbf{c} = (\mathbf{x} \cdot \mathbf{b} + \mathbf{y} \cdot \mathbf{b})\mathbf{c} = (\mathbf{x} \cdot \mathbf{b})\mathbf{c} + (\mathbf{y} \cdot \mathbf{b})\mathbf{c} = \mathbf{T}(\mathbf{x}) + \mathbf{T}(\mathbf{y})$$

and

$$\mathbf{T}(a\mathbf{x}) = ((a\mathbf{x}) \cdot \mathbf{b})\mathbf{c} = (a(\mathbf{x} \cdot \mathbf{b}))\mathbf{c} = a(\mathbf{x} \cdot \mathbf{b})\mathbf{c} = a\mathbf{T}(\mathbf{x})$$

for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^3$ and any $a \in \mathbf{R}$.

- (b) (5 points) Let

$$\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$, where T is the linear transformation defined in part (a).

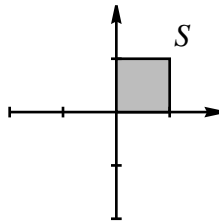
Solution. Since

$$\mathbf{T}(\mathbf{e}_1) = 2\mathbf{c} = \begin{bmatrix} -2 \\ 8 \end{bmatrix}, \quad \mathbf{T}(\mathbf{e}_2) = 3\mathbf{c} = \begin{bmatrix} -3 \\ 12 \end{bmatrix}, \quad \mathbf{T}(\mathbf{e}_3) = 5\mathbf{c} = \begin{bmatrix} -5 \\ 20 \end{bmatrix},$$

the matrix for \mathbf{T} is

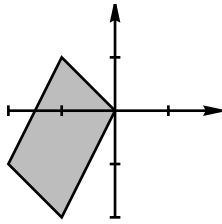
$$A = \begin{bmatrix} -2 & -3 & -5 \\ 8 & 12 & 20 \end{bmatrix}.$$

7. Let $S = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$.



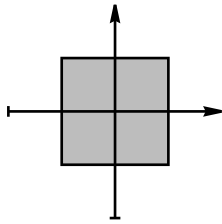
Determine whether or not each figure below is the image of S under some linear transformation. For those which are, find the matrix for such a transformation.

(a) (3 points)



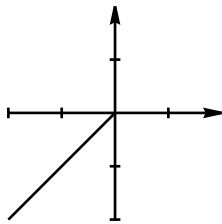
Solution. There are two possibilities, $\begin{bmatrix} -1 & -1 \\ 1 & -2 \end{bmatrix}$ or $\begin{bmatrix} -1 & -1 \\ -2 & 1 \end{bmatrix}$.

(b) (3 points)



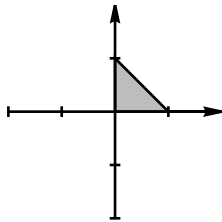
Solution. Not an image of S .

(c) (3 points)



Solution. There are several possibilities. One is $\begin{bmatrix} -2 & 0 \\ -2 & 0 \end{bmatrix}$.

(d) (3 points)



Solution. Not an image of S .

8. Let $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for \mathbf{R}^3 , and suppose that $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is a linear transformation satisfying

$$T(\mathbf{v}_1) = 2\mathbf{v}_3 \quad T(\mathbf{v}_2) = 2\mathbf{v}_2 \quad T(\mathbf{v}_3) = 2\mathbf{v}_1$$

(a) (6 points) Find the matrix B for T with respect to the basis β .

Solution.

$$B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(b) (6 points) Suppose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the matrix A for T with respect to the standard basis for \mathbf{R}^3 .

Solution. The change of basis matrix is

$$C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Its inverse is

$$C^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus

$$A = CBC^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 2 & 0 & -2 \\ 2 & -2 & 0 \end{bmatrix}.$$

9. (a) (5 points) Suppose A is an $n \times n$ matrix and that \mathbf{v} is an eigenvector of A with eigenvalue λ . Show that \mathbf{v} is an eigenvector of $A^2 + A$ with eigenvalue $\lambda^2 + \lambda$.

Solution. Applying $A^2 + A$ to \mathbf{v} , we get

$$\begin{aligned} (A^2 + A)\mathbf{v} &= A^2\mathbf{v} + A\mathbf{v} = A(A\mathbf{v}) + \lambda\mathbf{v} \\ &= A(\lambda\mathbf{v}) + \lambda\mathbf{v} = \lambda A\mathbf{v} + \lambda\mathbf{v} \\ &= \lambda^2\mathbf{v} + \lambda\mathbf{v} = (\lambda^2 + \lambda)\mathbf{v} \end{aligned}$$

so \mathbf{v} is an eigenvector of $A^2 + A$ with eigenvalue $\lambda^2 + \lambda$.

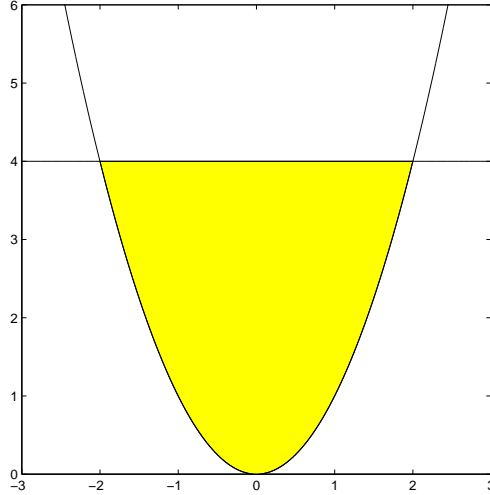
- (b) (5 points) Suppose A is a 3×3 matrix with eigenvalues -3 , -2 and 3 . Suppose $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Suppose further that f has a critical point at \mathbf{a} and that $Hf(\mathbf{a}) = A^2 + A$. Does f have a local maximum, a local minimum, or a saddle at \mathbf{a} ? Explain.

Solution. By the result of part (a), the eigenvalues of $A^2 + A$ are

$$\begin{aligned} (-3)^2 + (-3) &= 6 \\ (-2)^2 + (-2) &= 2 \\ (3)^2 + (3) &= 12, \end{aligned}$$

so $Hf(\mathbf{a})$ defines a positive-definite quadratic form, and therefore f has a local minimum at \mathbf{a} .

10. Let $D = \{(x, y) \in \mathbf{R}^2 \mid -2 \leq x \leq 2, x^2 \leq y \leq 4\}$



and let $f(x, y) = x^2y + y^2 - 4y$.

- (a) (5 points) Find all critical points of f in \mathbf{R}^2 , and identify which ones are in D .

Solution. Since

$$\frac{\partial f}{\partial x} = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 + 2y - 4,$$

the critical points of f are $(0, 2)$ and $(2, 0)$. Only $(0, 2)$ is in the domain D .

- (b) (5 points) Find the maximum and minimum values of f the line segment given by $\{(x, y) \mid y = 4, -2 \leq x \leq 2\}$.

Solution. Let $g(x) = f(x, 4) = 4x^2$. For $-2 \leq x \leq 2$ this function clearly has a minimum of 0 at $x = 0$ and a maximum of 16 at $x = -2$ and $x = 2$. Thus the maximum of f along the line segment is $f(-2, 4) = f(2, 4) = 16$ and the minimum of f along the line segment is $f(0, 4) = 0$.

- (c) (5 points) Find the maximum and minimum values of f on the parabolic arc given by $\{(x, y) \mid y = x^2, -2 \leq x \leq 2\}$.

Solution. Let $g(x) = f(x, x^2) = 2x^4 - 4x^2$. Since $g'(x) = 8x^3 - 8x = 8x(x^2 - 1)$, the critical points of g are $-1, 0$ and 1 . Since $g(-1) = g(1) = -2$, $g(0) = 0$ and $g(2) = g(-2) = 16$, the maximum of f on the arc is $f(2, 4) = f(-2, 4) = 16$ and the minimum of f on the arc is $f(-1, 1) = f(1, 1) = -2$.

- (d) (3 points) Find the maximum and minimum values of f on D .

Solution. Comparing the values from the previous two parts with $f(0, 2) = -4$, we see that the maximum of f on D is $f(2, 4) = f(-2, 4) = 16$ and the minimum of f on D is $f(0, 2) = -4$.

11. (10 points) The function $z(x, y)$ satisfies

$$x^2 + \frac{1}{2}y^4z + z^3 = 0,$$

and $z(3, 1) = -2$. Use implicit differentiation to compute

$$\left. \frac{\partial z}{\partial x} \right|_{(x,y)=(3,1)}.$$

Solution. Differentiating with respect to x gives

$$2x + \frac{1}{2}y^4 \frac{\partial z}{\partial x} + 3z^2 \frac{\partial z}{\partial x} = 0$$

At $(3, 1, -2)$ this becomes

$$6 + \frac{25}{2} \frac{\partial z}{\partial x} = 0,$$

so

$$\frac{\partial z}{\partial x} = -\frac{12}{25}.$$

12. (10 points) Define $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ by

$$f(x, y, z) = x^2 + y^3 + z^4.$$

Consider the level surface in \mathbf{R}^3 ,

$$S = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = 18\}.$$

Find the equation for the tangent plane to S at the point $(3, 2, 1)$.

Solution. Since

$$\nabla f(x, y, z) = (2x, 3y^2, 4z^3) \implies \nabla f(3, 2, 1) = (6, 12, 4),$$

the equation of the tangent plane is $6(x - 3) + 12(y - 2) + 4(z - 1) = 0$.

13. (10 points) Define $\mathbf{f} : \mathbf{R} \rightarrow \mathbf{R}^3$ by $\mathbf{f}(t) = (1, t, t^2)$. Suppose $g : \mathbf{R}^3 \rightarrow \mathbf{R}$ satisfies

$$\frac{\partial g}{\partial x}(1, 2, 4) = 5, \quad \frac{\partial g}{\partial y}(1, 2, 4) = 6, \quad \frac{\partial g}{\partial z}(1, 2, 4) = 7.$$

Calculate

$$\left. \frac{d}{dt} g(\mathbf{f}(t)) \right|_{t=2}.$$

Solution. Since $\mathbf{f}'(t) = (0, 1, 2t)$, the Chain Rule implies

$$\left. \frac{d}{dt} g(\mathbf{f}(t)) \right|_{t=2} = Jg(\mathbf{f}(2))J\mathbf{f}'(2) = Jg(1, 2, 4)\mathbf{f}'(2) = [5 \ 6 \ 7] \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} = 34.$$

14. Let $f(x, y) = x^2 - 2x + y^2 - 6y$.

- (a) (5 points) Find all critical points of f .

Solution. Since

$$\frac{\partial f}{\partial x} = 2x - 2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y - 6,$$

the only critical point of f is $(1, 3)$.

- (b) (7 points) Use Lagrange multipliers to find the maximum and minimum of f on the circle $\{(x, y) \mid x^2 + y^2 = 40\}$.

Solution. The Lagrange multiplier equations are

$$\begin{aligned} 2x - 2 &= 2\lambda x \\ 2y - 6 &= 2\lambda y \\ x^2 + y^2 &= 40. \end{aligned}$$

Multiplying the first equation by y and the second by x and subtracting gives $y = 3x$. Thus the third equation becomes $10x^2 = 40$, so $x = 2$ or $x = -2$. Since $f(2, 6) = 0$ and $f(-2, -6) = 80$, the maximum of f on the circle is 80 and the minimum of f on the circle is 0.

- (c) (3 points) Find the maximum and minimum of f on the disk $\{(x, y) \mid x^2 + y^2 \leq 40\}$.

Solution. Since $f(1, 3) = -10$, the maximum of f on the disk is $f(-2, -6) = 80$ and the minimum of f on the disk is $f(1, 3) = -10$.

15. (a) (5 points) Let $f(x, y) = \cos x + 5xe^y + 3y^2 + x^3$. Find the Hessian of f at $(0, 0)$.

Solution. Since

$$\frac{\partial f}{\partial x} = -\sin x + 5e^y + 3x^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 5xe^y + 6y$$

the Hessian of f is

$$Hf(x, y) = \begin{bmatrix} -\cos x + 6x & 5e^y \\ 5e^y & 5xe^y + 6 \end{bmatrix}.$$

- (b) (5 points) Suppose that $f : \mathbf{R}^3 \rightarrow \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Let \mathbf{p} be a critical point of f and suppose that the Hessian of f at \mathbf{p} is

$$Hf(\mathbf{p}) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Does f have a local maximum, local minimum, or saddle at \mathbf{p} ?

Solution. The eigenvalues of $Hf(\mathbf{p})$ are -1 , -2 and -3 , so $Hf(\mathbf{p})$ defines a negative-definite quadratic form, and therefore f has a local maximum at \mathbf{p} .

- (c) (5 points) Suppose that $g : \mathbf{R}^3 \rightarrow \mathbf{R}$ is a function whose second-order partial derivatives are continuous. Let \mathbf{q} be a critical point of g and suppose that the Hessian of g at \mathbf{q} is

$$Hg(\mathbf{q}) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Does g have a local maximum, local minimum, or saddle at \mathbf{q} ?

Solution. The eigenvalues of $Hg(\mathbf{q})$ are 3 and -1 , so $Hg(\mathbf{q})$ defines an indefinite quadratic form, and therefore g has a saddle at \mathbf{q} .

16. Let $f : \mathbf{R}^2 \rightarrow \mathbf{R}$ be defined by $f(x, y) = xy^2 - x^3$.

(a) (6 points) What is the direction of greatest *decrease* of f at $(1, 1)$? Express your answer as a unit vector.

Solution. The direction of greatest decrease is in the direction of minus the gradient. Since

$$\nabla f(x, y) = (y^2 - 3x^2, 2xy) \implies \nabla f(1, 1) = (-2, 2),$$

the direction of greatest decrease is

$$\mathbf{u} = -\frac{\nabla f(1, 1)}{\|\nabla f(1, 1)\|} = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

(b) (6 points) What is the directional derivative of f at the point $(1, 2)$ in the direction toward the point $(4, 3)$?

Solution. The direction from $(1, 2)$ toward $(4, 3)$ is

$$\mathbf{u} = \frac{1}{\sqrt{10}}(3, 1)$$

so

$$D_{\mathbf{u}}f(1, 2) = \nabla f(1, 2) \cdot \mathbf{u} = (1, 4) \cdot \mathbf{u} = \frac{7}{\sqrt{10}}.$$