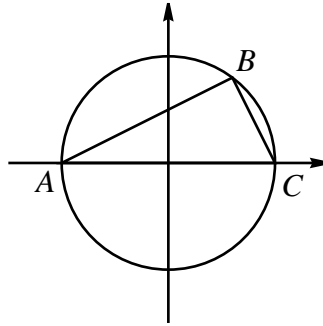


MATH 51 FINAL EXAM (AUTUMN 2000)

1. Let

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \quad \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix}$$

- (a) (6 points) Find the dimension of  $\text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$ .
- (b) (8 points) Find all vectors  $\mathbf{v}$  which are simultaneously orthogonal (i.e. perpendicular) to all three vectors  $\mathbf{u}_1, \mathbf{u}_2$  and  $\mathbf{u}_3$ .
2. (10 points) Suppose  $B = (x, y)$  is a point on the circle of radius 1 centered at the origin. That is,  $x$  and  $y$  satisfy  $x^2 + y^2 = 1$ . Let  $A = (-1, 0)$ ,  $C = (1, 0)$  and assume  $y \neq 0$  (so that  $B$  is not equal to  $A$  or  $C$ ).



Use dot products to show that angle  $ABC$  is a right angle.

3. Suppose  $A$  is a  $5 \times 5$  matrix with

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

For each part below, give the answer when possible. Otherwise answer “not enough information”.

- (a) (2 points) Find a basis for  $N(A)$ .
- (b) (2 points) Find  $\dim(N(A))$ .
- (c) (2 points) Find a basis for  $C(A)$ .
- (d) (2 points) Find  $\dim(C(A))$ .
- (e) (2 points) Find the rank of  $A$ .
- (f) (2 points) Find a vector  $\mathbf{b} \in \mathbf{R}^5$  such that  $A\mathbf{x} = \mathbf{b}$  has no solutions.
- (g) (2 points) Are there vectors  $\mathbf{b} \in \mathbf{R}^5$  such that  $A\mathbf{x} = \mathbf{b}$  has *exactly* one solution?

(h) (2 points) Find the eigenvalues of  $A$ .

4. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{bmatrix}$$

(a) (5 points) Compute  $\det(A)$ .

(b) (7 points) Find  $A^{-1}$ .

5. (a) (6 points) Find the eigenvalues of  $A$ .

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 7 & 8 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) (8 points) Let

$$B = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -4 & 1 \end{bmatrix}$$

$\lambda = 3$  is an eigenvalue of  $B$  (you do not need to verify this). Find a basis for the eigenspace  $E_3 = \{\mathbf{v} \in \mathbf{R}^3 \mid B\mathbf{v} = 3\mathbf{v}\}$ .

6. (a) (5 points) Show that, for each choice of fixed vectors  $\mathbf{b} \in \mathbf{R}^3$  and  $\mathbf{c} \in \mathbf{R}^2$ , the formula

$$T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{b})\mathbf{c}$$

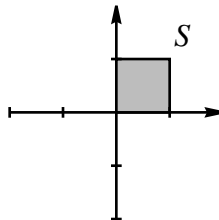
defines a linear transformation  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^2$ .

(b) (5 points) Let

$$\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad \mathbf{c} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

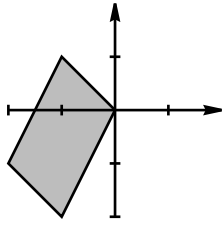
Find the matrix  $A$  such that  $T(\mathbf{x}) = A\mathbf{x}$ , where  $T$  is the linear transformation defined in part (a).

7. Let  $S = \{(x, y) \in \mathbf{R}^2 \mid 0 \leq x \leq 1, 0 \leq y \leq 1\}$ .

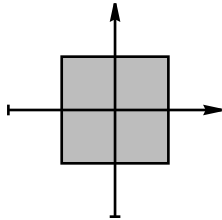


Determine whether or not each figure below is the image of  $S$  under some linear transformation. For those which are, find the matrix for such a transformation.

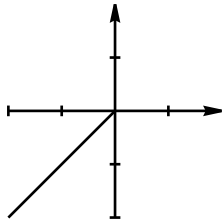
(a) (3 points)



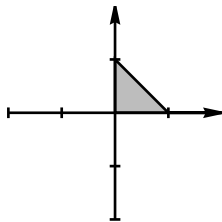
(b) (3 points)



(c) (3 points)



(d) (3 points)



8. Let  $\beta = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  be a basis for  $\mathbf{R}^3$ , and suppose that  $T : \mathbf{R}^3 \rightarrow \mathbf{R}^3$  is a linear transformation satisfying

$$T(\mathbf{v}_1) = 2\mathbf{v}_3 \quad T(\mathbf{v}_2) = 2\mathbf{v}_2 \quad T(\mathbf{v}_3) = 2\mathbf{v}_1$$

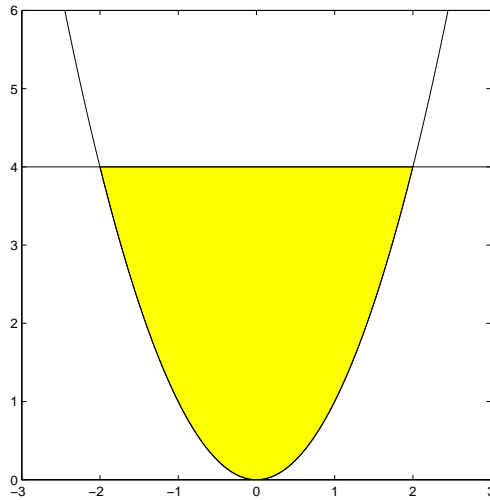
(a) (6 points) Find the matrix  $B$  for  $T$  with respect to the basis  $\beta$ .

(b) (6 points) Suppose

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Find the matrix  $A$  for  $T$  with respect to the standard basis for  $\mathbf{R}^3$ .

9. (a) (5 points) Suppose  $A$  is an  $n \times n$  matrix and that  $\mathbf{v}$  is an eigenvector of  $A$  with eigenvalue  $\lambda$ . Show that  $\mathbf{v}$  is an eigenvector of  $A^2 + A$  with eigenvalue  $\lambda^2 + \lambda$ .
- (b) (5 points) Suppose  $A$  is a  $3 \times 3$  matrix with eigenvalues  $-3$ ,  $-2$  and  $3$ . Suppose  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  is a function whose second-order partial derivatives are continuous. Suppose further that  $f$  has a critical point at  $\mathbf{a}$  and that  $Hf(\mathbf{a}) = A^2 + A$ . Does  $f$  have a local maximum, a local minimum, or a saddle at  $\mathbf{a}$ ? Explain.
10. Let  $D = \{(x, y) \in \mathbf{R}^2 \mid -2 \leq x \leq 2, x^2 \leq y \leq 4\}$



and let  $f(x, y) = x^2y + y^2 - 4y$ .

- (a) (5 points) Find all critical points of  $f$  in  $\mathbf{R}^2$ , and identify which ones are in  $D$ .
- (b) (5 points) Find the maximum and minimum values of  $f$  the line segment given by  $\{(x, y) \mid y = 4, -2 \leq x \leq 2\}$ .
- (c) (5 points) Find the maximum and minimum values of  $f$  on the parabolic arc given by  $\{(x, y) \mid y = x^2, -2 \leq x \leq 2\}$ .
- (d) (3 points) Find the maximum and minimum values of  $f$  on  $D$ .
11. (10 points) The function  $z(x, y)$  satisfies

$$x^2 + \frac{1}{2}y^4z + z^3 = 0,$$

and  $z(3, 1) = -2$ . Use implicit differentiation to compute

$$\left. \frac{\partial z}{\partial x} \right|_{(x,y)=(3,1)}.$$

12. (10 points) Define  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  by

$$f(x, y, z) = x^2 + y^3 + z^4.$$

Consider the level surface in  $\mathbf{R}^3$ ,

$$S = \{(x, y, z) \in \mathbf{R}^3 \mid f(x, y, z) = 18\}.$$

Find the equation for the tangent plane to  $S$  at the point  $(3, 2, 1)$ .

13. (10 points) Define  $\mathbf{f} : \mathbf{R} \rightarrow \mathbf{R}^3$  by  $f(t) = (1, t, t^2)$ . Suppose  $g : \mathbf{R}^3 \rightarrow \mathbf{R}$  satisfies

$$\frac{\partial g}{\partial x}(1, 2, 4) = 5, \quad \frac{\partial g}{\partial y}(1, 2, 4) = 6, \quad \frac{\partial g}{\partial z}(1, 2, 4) = 7.$$

Calculate

$$\left. \frac{d}{dt}g(\mathbf{f}(t)) \right|_{t=2}.$$

14. Let  $f(x, y) = x^2 - 2x + y^2 - 6y$ .

- (a) (5 points) Find all critical points of  $f$ .
- (b) (7 points) Use Lagrange multipliers to find the maximum and minimum of  $f$  on the circle  $\{(x, y) \mid x^2 + y^2 = 40\}$ .
- (c) (3 points) Find the maximum and minimum of  $f$  on the disk  $\{(x, y) \mid x^2 + y^2 \leq 40\}$ .
15. (a) (5 points) Let  $f(x, y) = \cos x + 5xe^y + 3y^2 + x^3$ . Find the Hessian of  $f$  at  $(0, 0)$ .
- (b) (5 points) Suppose that  $f : \mathbf{R}^3 \rightarrow \mathbf{R}$  is a function whose second-order partial derivatives are continuous. Let  $\mathbf{p}$  be a critical point of  $f$  and suppose that the Hessian of  $f$  at  $\mathbf{p}$  is

$$Hf(\mathbf{p}) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Does  $f$  have a local maximum, local minimum, or saddle at  $\mathbf{p}$ ?

- (c) (5 points) Suppose that  $g : \mathbf{R}^3 \rightarrow \mathbf{R}$  is a function whose second-order partial derivatives are continuous. Let  $\mathbf{q}$  be a critical point of  $g$  and suppose that the Hessian of  $g$  at  $\mathbf{q}$  is

$$Hg(\mathbf{q}) = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Does  $g$  have a local maximum, local minimum, or saddle at  $\mathbf{q}$ ?

16. Let  $f : \mathbf{R}^2 \rightarrow \mathbf{R}$  be defined by  $f(x, y) = xy^2 - x^3$ .

- (a) (6 points) What is the direction of greatest *decrease* of  $f$  at  $(1, 1)$ ? Express your answer as a unit vector.
- (b) (6 points) What is the directional derivative of  $f$  at the point  $(1, 2)$  in the direction toward the point  $(4, 3)$ ?