Math 42 Final Exam — March 14, 2011

Name: ______________________________ **SUID#:** ______________

Circle your section:				
Jack Hall 05 (11-11:50am) 10 (2:15-3:05pm)	Vitaly Katsnelson 04 (11-11:50am) 07 (1:15-2:05pm)	Maks Radziwill 02 (10-10:50am) 09 (2:15-3:05pm)	David Sher 03 (10-10:50am) 08 (1:15-2:05pm)	Tracy Nance ACE

- Complete the following problems. In order to receive full credit, please show all of your work and justify your answers.
- You do not need to simplify your answers unless specifically instructed to do so. You may use any result from class that you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- **You have 3 hours.** This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted. If you finish early, you must hand your exam paper to a member of teaching staff.
- Please check that your copy of this exam contains 17 numbered pages and is correctly stapled.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

> "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ______________________________

The following boxes are strictly for grading purposes. Please do not mark.

Question:	1	2	3	4	5	6	7	8	9	10	11	**Total**
Points:	10	12	12	10	12	12	15	12	10	13	14	132
Score:												

1. (10 points) Let R be the region in the xy-plane below the curve $y = \dfrac{1}{1+x^2}$ and above the line $y = \dfrac{1}{2}$.

   (a) Suppose S_1 is the solid generated by rotating R about the x-axis. Set up two different integrals, each in terms of a single variable, representing the volume of S_1. Cite the method used in each case, but don't evaluate either integral.

   (b) Suppose S_2 is the solid whose base is R and whose cross-sections perpendicular to the x-axis are all squares. Set up an integral representing the volume of S_2, but don't evaluate the integral.

2. (12 points)

(a) Evaluate $\int_1^e \frac{1}{x \ln x}\, dx$ or explain why its value does not exist; show all reasoning.

(b) Determine all values of p for which the integral $\int_1^e \frac{1}{x(\ln x)^p}\, dx$ converges, and evaluate the integral for those values of p.

3. (12 points) For this problem, use the following information:

- If f is a normal ("bell-shaped" or "Gaussian") probability density function, then f has the general form

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}$$

- A partial list of approximate values of the function

$$P(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt \quad \text{is given at right:}$$

$P(0.25) \approx 0.60$	$P(1.75) \approx 0.960$
$P(0.5) \approx 0.69$	$P(2.0) \approx 0.977$
$P(0.75) \approx 0.77$	$P(2.25) \approx 0.988$
$P(1.0) \approx 0.84$	$P(2.5) \approx 0.994$
$P(1.25) \approx 0.89$	$P(2.75) \approx 0.997$
$P(1.5) \approx 0.93$	$P(3.0) \approx 0.999$

(a) A subatomic particle is in the ground state of a quantum harmonic oscillator; its position x along a line is given by the normal probability density function

$$f_0(x) = \frac{1}{2\sqrt{2\pi}} e^{-x^2/8}$$

What is the approximate probability that the particle's position is in the interval $[-1, 0]$? Your answer should be a number; justify it by writing an integral expression that represents this probability and showing how to find its value.

(b) Use integration by parts to show that

$$\int x^2 e^{-x^2/8}\,dx = -4xe^{-x^2/8} + \int 4e^{-x^2/8}\,dx$$

(c) The particle absorbs a photon and transitions to the next energy level. The probability density function for the particle's position now becomes

$$f_1(x) = \frac{x^2}{8\sqrt{2\pi}} e^{-x^2/8}$$

Use the information provided in part (b) and in the preceding chart to compute the approximate probability that the particle is now in the interval $[0, 1.5]$. (Give your answer as a expression involving only numbers, but do not simplify your answer.)

4. (10 points) Determine whether each of the series below converges or diverges. Indicate clearly which tests you use and how you apply them.

   (a) $\sum_{n=1}^{\infty} \frac{1}{3^n - 2^n}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(n+1)!}$

5. (12 points) For each of the following questions, give an example of a power series in x with the specified property or properties; for full credit, your answer should be given using sigma notation. *You do not have to justify your answers.* (Please treat each part as independent from the others; properties do not carry over from one part to the next.)

   (a) The power series is centered at $x = -1$.

   (b) The power series is centered at $x = 0$ and has radius of convergence ∞.

   (c) The power series is centered at $x = 2$ and has radius of convergence equal to 3.

   (d) The power series has interval of convergence $[2, 6)$.

6. (12 points) Consider the function $f(x) = e^x \cos x$.

   (a) Find $T_3(x)$, the third-degree Taylor polynomial for f centered at 0. Show all the steps of your computation.

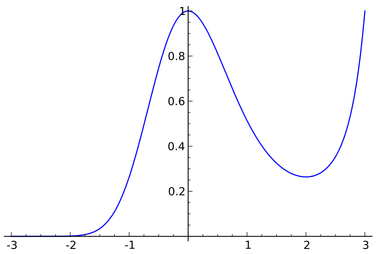
   (b) Use T_3 to obtain an approximation for $e^{0.1} \cos(0.1)$. (You do not need to simplify your answer.)

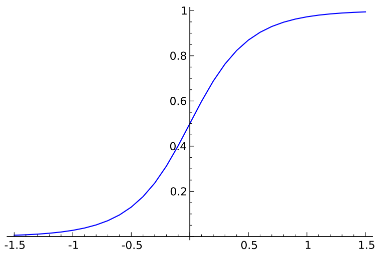
(c) Compute the 4th derivative $f^{(4)}(x)$ of $f(x)$ and show that

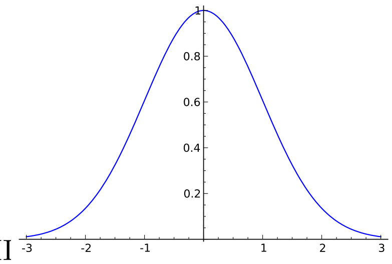
$$f^{(4)}(x) = -4f(x) \quad \text{for all } x.$$

(d) Use the statement of part (c) to show that the estimate of part (b) is accurate to within 5×10^{-5}. Precisely cite the steps of your reasoning. (You may use the fact that $2 \leq e \leq 3$.)

7. (15 points) Each of the curves below is a solution to exactly one differential equation in the chart at bottom. Match each curve with its equation; no justification is necessary.

I


II


III


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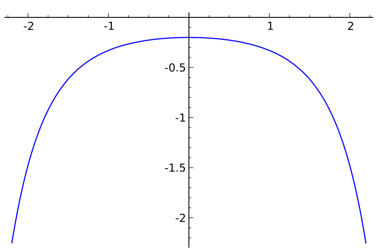

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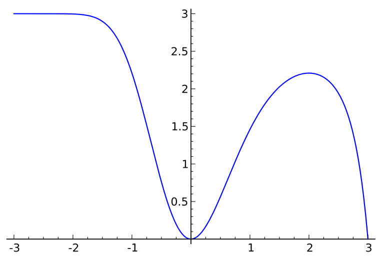
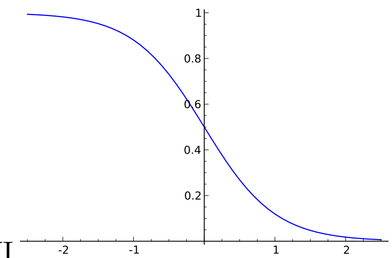
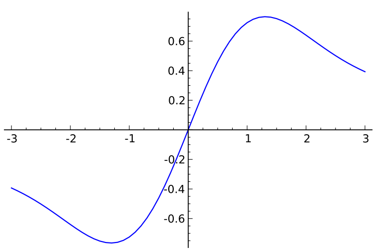
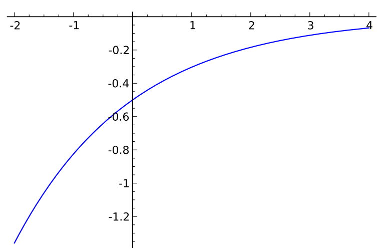


VI


VII


VIII


Equation	I, II, III, IV, V, VI, VII, or VIII	Equation	I, II, III, IV, V, VI, VII, or VIII
$y' = (y-3)x(x-2)$		$y' = -xy$	
$y' = 1 - xy$		$y' = -\frac{y}{2}$	
$y' = 2y(y-1)$		$y' = x(x-2)y$	
$y' = \sin(\pi y)$		$y' = xy$	

8. (12 points)

(a) Solve the initial value problem

$$\frac{dy}{dx} = x^2 e^{y+x}, \quad y(0) = 0.$$

(b) Solve the initial value problem

$$y\,\frac{dy}{dx} + 2x = 0, \quad y(0) = -3.$$

9. (10 points) The air in a room with volume $500\,\text{m}^3$ is permeated with a toxic substance. Let $x(t)$ denote the concentration, in mg/m^3, of the toxic substance in the room after t minutes, and suppose that $x(0) = 10^{-4}\,\text{mg}/\text{m}^3$. A scrubbing system removes air from the room at a rate of $2\,\text{m}^3/\text{min}$, *partially* cleans it, and quickly reintroduces it into the room at the same rate. Assume that at any time t, the concentration of the toxic substance in the cleaner air that is being reintroduced into the room is equal to $x(t)^{3/2}$.

(a) Write a differential equation for x.

(b) Determine the equilibrium solutions of the differential equation from part (a).

(c) Use Euler's method with a step size of 1 minute to estimate $x(2)$, the concentration after 2 minutes. Show your steps, but you do *not* need to simplify your answer.

10. (13 points) A population $P(t)$ (measured in thousands of beings) is roughly described at time t by a logistic model with proportionality constant $k = 1$ and carrying capacity 2.

(a) Write a differential equation for P.

(b) Determine the equilibrium solutions of the equation in (a).

(c) Solve the initial value problem given by the initial condition $P(0) = 1$. What is the long-term behavior of the population?

(d) A severe disease appears among the population. The mortality due to the disease is approximately constant and equal to 0.5 thousands of beings per unit time. Modify the differential equation from (a) to account for the increased mortality.

(e) Solve the differential equation from (d), subject to the initial condition $P(0) = 2$. What is the long-term behavior of the population subject to the disease?

11. (14 points) Two populations coexist within a territory, and are approximately modeled by the system of differential equations

$$\begin{cases} x' = 4x - xy \\ y' = -y + \frac{xy}{2} \end{cases}$$

(a) Describe the nature of the relationship between the two species: is it one of competition, cooperation, or predator and prey, and how can you tell? (If the relationship is predator and prey, make sure to explain how to tell which species is which.)

(b) For each species, describe what happens if the other is not present.

(c) Find all equilibrium solutions of this system.

For easy reference, here again is the system:

$$\begin{cases} x' = 4x - xy \\ y' = -y + \frac{xy}{2} \end{cases}$$

(d) Suppose that at time $t = 0$, we have $x(0) = 2$ and $y(0) = 2$. For each of the two populations, determine if it is increasing, decreasing, or not changing size at this moment in time, and explain how you know.

(e) In the direction field below, mark the equilibrium solutions found in part (c). Furthermore, draw an approximation to the phase trajectory of the solution with initial condition $x(0) = 2$, $y(0) = 2$. Mark with an arrow the direction in which the phase trajectory is traversed.