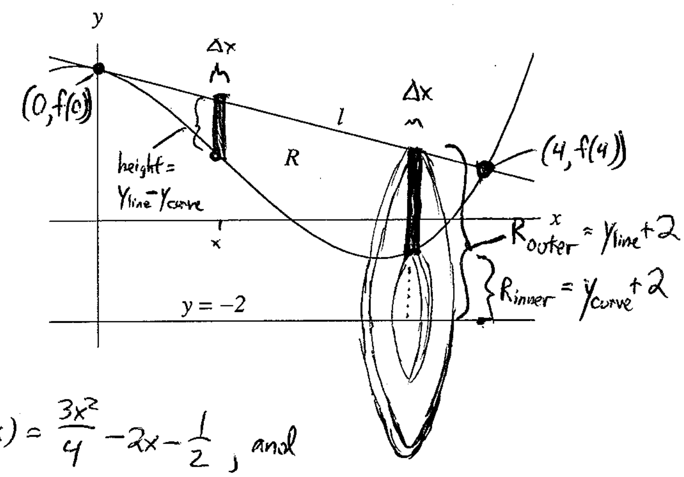
SOLUTIONS

1. (10 points)



Let f be the function given by

$$f(x) = \frac{x^3}{4} - x^2 - \frac{x}{2} + 3.$$

The line l is tangent to the graph of f at $x = 0$. Let R be the region bounded by f and l.

(a) Find the equation of the tangent line l.

l has slope equal to $f'(0)$: $f'(x) = \frac{3x^2}{4} - 2x - \frac{1}{2}$, and so $f'(0) = -\frac{1}{2}$. Line l passes thru $(0, f(0)) = (0,3)$.

$\Rightarrow$ l has equation $y-3 = -\frac{1}{2}(x-0) \Rightarrow \boxed{y = -\frac{1}{2}x+3}$.

(b) Find the area of R.

Line l intersects curve when $\frac{x^3}{4} - x^2 - \frac{1}{2}x + 3 = -\frac{1}{2}x+3$, so $\frac{x^3}{4} - x^2 = 0$,

i.e. when $x^2\left(\frac{x}{4}-1\right)=0$; thus, $x=0$ or $x=4$.

Slicing R (from $x=0$ to $x=4$) into vertical slices of width Δx yields pieces that can be approximated by rectangles of height $y_{line} - y_{curve}$; slice at coord. x has height $= \left(-\frac{1}{2}x+3\right) - \left(\frac{x^3}{4} - x^2 - \frac{x}{2} + 3\right) = x^2 - \frac{x^3}{4}$. Thus area $\approx \sum (\text{height})\Delta x$, so letting $\Delta x \to 0$, we find Area $= \int_{x=0}^{x=4} \left(x^2 - \frac{x^3}{4}\right)dx = \frac{x^3}{3} - \frac{x^4}{16}\Big]_0^4 = \boxed{\frac{4^3}{3} - \frac{4^4}{16} = \frac{16}{3}}$.

(c) Write an integral for the volume of the solid generated when R is rotated about the line $y = -2$. You do not need to evaluate this integral.

Vertical slices of width Δx are rotated, becoming washers. Slice at coord. x becomes a washer with inner radius $y_{curve} - (-2) = f(x)+2 = \frac{x^3}{4} - x^2 - \frac{x}{2} + 5$, and outer radius $y_{line} - (-2) = -\frac{1}{2}x + 3 + 2 = -\frac{1}{2}x+5$; thus the volume of the approximating washer is $\text{Area}(x)\cdot\Delta x = (\pi R_{outer}^2 - \pi R_{inner}^2)\Delta x = \pi\left(\left(-\frac{1}{2}x+5\right)^2 - \left(\frac{x^3}{4} - x^2 - \frac{x}{2}+5\right)^2\right)\Delta x$,

so letting $\Delta x \to 0$ we obtain $\boxed{\text{Volume} = \int_0^4 \pi\left(\left(-\frac{1}{2}x+5\right)^2 - \left(\frac{x^3}{4} - x^2 - \frac{x}{2}+5\right)^2\right)dx}$.

2. (10 points) Given a continuous function $f(x)$ on the interval $[a, b]$. Suppose that we wish to approximate the integral

$$\int_a^b f(x)\,dx$$

using one of the basic approximation techniques (Midpoint Rule, Trapezoidal Rule, Simpson's Rule). Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

T **F** The Midpoint Rule always produces a more accurate approximation than the Trapezoidal Rule (for a fixed number of subintervals).

All you need is one counterexample, and here's one: take $n=1$ and consider $\int_0^{3\pi} \sin x\,dx$.

T **F** The smaller the value of K_2 that we choose, the more accurate our Midpoint Rule approximation will be (for a fixed number of subintervals).

K_2 has no impact on value of error; it only affects the estimate of the error's "worst-case" size. (Plus, you can't "choose" a value of K_2 any smaller than the maximum of $|f''(x)|$ on $[a,b]$!)

T **F** If $f(x)$ is a polynomial function of degree 3, then Simpson's Rule always produces the *actual* value of the integral.

If f is degree 3, then notice $f^{(4)}(x)=0$ for all x. Thus we can take $K_4=0$, which means that the Error Bound Formula tells us that $|\text{error}| \le 0$, meaning error $=0$!

T **F** If $f(x)$ is increasing on the interval $[a, b]$, then the Midpoint Rule always gives an underestimate of the actual value of the integral.

M_n sometimes gives an overest, sometimes gives underest, for increasing f. (To see this, consider increasing functions of varying concavities.)

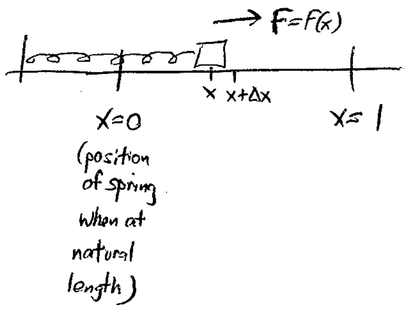
T **F** If $f(x)$ is concave down on the interval $[a, b]$, then the Trapezoidal Rule always gives an overestimate of the actual value of the integral.

The relevant picture is:

(So fact T_n never gives an overestimate!)

3. (5 points) A spring is tested for various properties. It is found to obey Hooke's Law, and it is determined that the work required to stretch it 1 ft beyond its natural length is 12 ft-lb. How much work is needed to stretch it $\frac{3}{4}$ ft beyond its natural length?

(*Hooke's Law* states that the force required to hold a spring in a given position is proportional to the distance that the spring is stretched from its natural length; that is, if x represents this latter amount, then the force $F = kx$ for some constant k.)



In stretching the spring 1 ft beyond its natural length, we are exerting a variable force $F(x)=kx$ from $x=0$ to $x=1$ (where x = amount that the spring is stretched beyond its natural length).

The work required to do this may be approximated by a sum of small pieces of work: break the interval from $x=0$ to $x=1$ into pieces of width Δx, and note that if Δx is small, then the force exerted between x and $x+\Delta x$ is approximately constant, nearly equal to $F(x)$. Thus the work required to move the spring from coord. x to coord. $x+\Delta x$ is approximately (force)(distance) $= F(x)\Delta x$. It follows that:

$$\text{total work} \approx \sum_{(\text{pieces})} F(x)\Delta x \quad , \text{ which is a Riemann sum,}$$

and we may use an integral in place of this sum to obtain the exact work (i.e. letting $\Delta x \to 0$):

$$\text{Work} = \int_{x=0}^{x=1} F(x)\,dx = \int_0^1 kx\,dx = \frac{kx^2}{2}\Big]_{x=0}^{x=1} = \frac{k}{2}.$$

But we know this equals 12 ft-lb, so $\boxed{k=24}$. Thus, the work required to stretch the spring from $x=0$ to $x=3/4$ is:

$$W = \int_0^{3/4} F(x)\,dx = \frac{24x^2}{2}\Big]_0^{3/4} = \boxed{\frac{27}{4}\text{ ft.lb}}.$$

4. (12 points) The time gaps between consecutive bursts of solar particles onto a certain detector is found to be closely modeled by the following probability density function:

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ \dfrac{C}{(4+x^2)^{3/2}} & \text{if } x \geq 0, \end{cases}$$

where C is a positive constant. Complete the following, giving full justification:

(a) Find C, given that f is a probability density function.

Since f is a PDF, we require $1 = \int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \frac{C}{(4+x^2)^{3/2}}\,dx = C\int_0^{\infty} \frac{dx}{(4+x^2)^{3/2}}$.

Now $\int_0^{\infty} \frac{dx}{(4+x^2)^{3/2}} = \lim_{N\to\infty} \int_0^{N} \frac{dx}{(4+x^2)^{3/2}} = \lim_{N\to\infty} \int_0^{\arctan N/2} \frac{2\sec^2\theta\,d\theta}{(4+4\tan^2\theta)^{3/2}} = \lim_{N\to\infty} \int_0^{\arctan N/2} \frac{2\sec^2\theta\,d\theta}{(4\sec^2\theta)^{3/2}}$

$$\left\{ \begin{array}{l} x = 2\tan\theta \\ dx = 2\sec^2\theta\,d\theta \\ x=0 \Rightarrow \theta = 0 \\ x = N \Rightarrow \theta = \arctan\frac{N}{2} \end{array} \right\}$$

$= \lim_{N\to\infty} \frac{2}{8}\int_0^{\arctan N/2} \cos\theta\,d\theta$

$= \lim_{N\to\infty} \frac{1}{4}\sin\theta\Big]_0^{\arctan N/2}$

$= \lim_{N\to\infty} \frac{1}{4}\sin\left(\arctan\frac{N}{2}\right) = \frac{1}{4}\sin\left(\frac{\pi}{2}\right) = \frac{1}{4}$,

where we used the fact that $\lim_{x\to\infty} \arctan x = \frac{\pi}{2}$. Thus, $1 = C\cdot\frac{1}{4}$, so that $\boxed{C=4}$.

(b) Find the mean time gap.

We have $\mu = \int_{-\infty}^{\infty} x f(x)\,dx = \int_0^{\infty} \frac{Cx}{(4+x^2)^{3/2}}\,dx = \lim_{N\to\infty}\int_0^N \frac{4x}{(4+x^2)^{3/2}}\,dx$.

Let $\left\{ \begin{array}{l} u = 4+x^2 \\ du = 2x\,dx \end{array} \right\}$, so that $\left\{ \begin{array}{l} x=0 \Rightarrow u=4 \\ x=N \Rightarrow u = 4+N^2 \end{array} \right\}$, and thus

$$\mu = \lim_{N\to\infty}\int_0^N \frac{4x}{(4+x^2)^{3/2}}\,dx = \lim_{N\to\infty}\int_4^{4+N^2} \frac{2\,du}{u^{3/2}} = \lim_{N\to\infty}\left[\frac{2u^{-1/2}}{-1/2}\right]_4^{4+N^2}$$

$$= \lim_{N\to\infty} 4\left(\frac{1}{\sqrt{4}} - \frac{1}{\sqrt{4+N^2}}\right) = 4\left(\frac{1}{\sqrt{4}} - 0\right) = \boxed{2}.$$

(Note: no trig substitution is needed for this part!)

5. (9 points)

(a) Find (with justification) all values of k such that the function $y = e^{kx}$ is a solution of the differential equation
$$\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = 0.$$

If $y=e^{kx}$, then $\frac{dy}{dx} = ke^{kx}$ and $\frac{d^2y}{dx^2} = k\cdot(ke^{kx}) = k^2e^{kx}$.

Thus if $y=e^{kx}$ satisfies the differential equation, we have

$$k^2e^{kx} - 7(ke^{kx}) + 6(e^{kx}) = 0,$$ i.e.

$$(k^2-7k+6)e^{kx} = 0$$ (as functions, i.e. for all x).

The function on the left can only be equal to the constant function 0 when $k^2-7k+6=0$, i.e. when $(k-6)(k-1)=0$; thus, $\boxed{k=1 \text{ and } k=6}$ are the only values of k that work.

(b) Which of the following families of functions is the solution to the differential equation $\frac{dy}{dt} = 3y+1$? (Here C stands for any constant.) No justification is necessary; just circle your answer.

(i) $y = Ce^{3t} + t$

(iii) $y = e^{3t} + t + C$

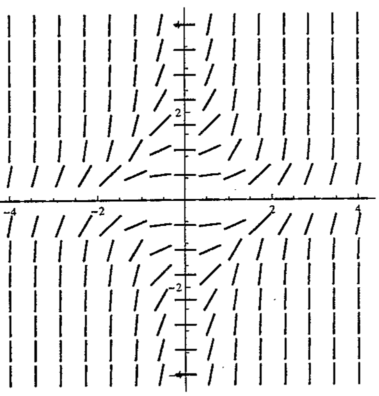
(ii) $y = e^{3t} - \frac{1}{3} + C$

(iv) $y = Ce^{3t} - \frac{1}{3}$

(Quick way to check: for each y, differentiate and compare to $3y+1$.)

(You could also solve the equation by separating variables, etc.)

6. (15 points) Match the direction fields below with their differential equations. (The horizontal variable is t; the vertical is y.) Also indicate which two equations do not have matches.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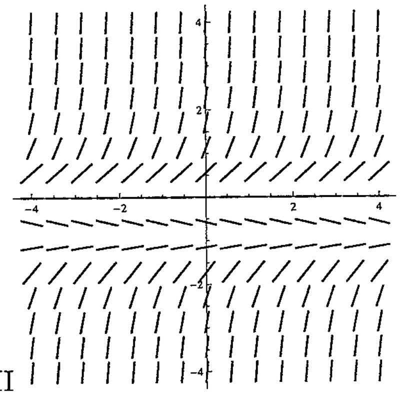
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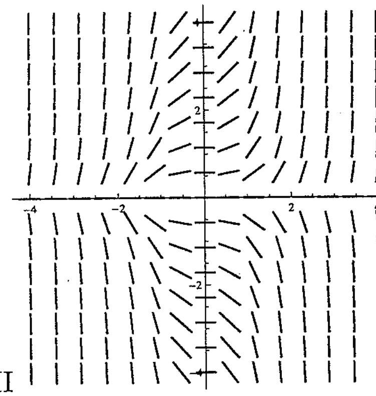
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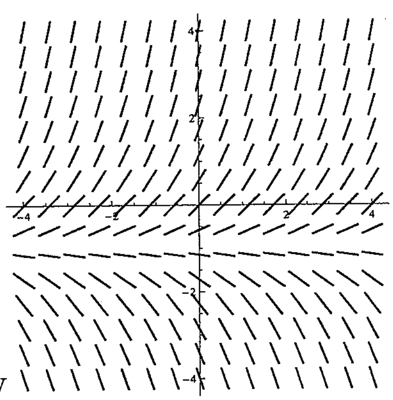
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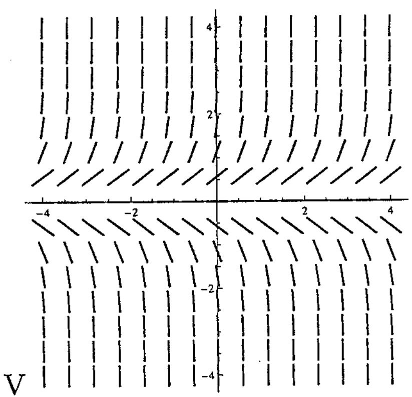
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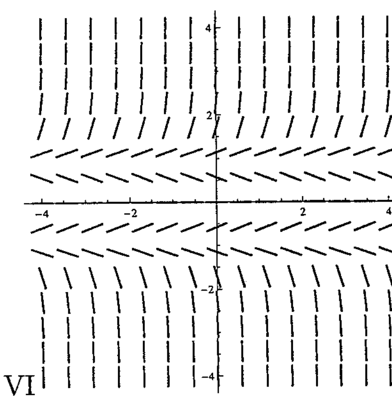

VI

Equation	I, II, III, IV, V VI, or "none"	Equation	I, II, III, IV, V VI, or "none"
$dy/dt = ty^2$	none	$dy/dt = t^2y^2$	I
$dy/dt = t^2y$	III	$dy/dt = y(y^2-1)$	VI
$dy/dt = y+t$	none	$dy/dt = y(y^2+1)$	V
$dy/dt = 1+y$	IV	$dy/dt = y(y+1)$	II

7. (11 points) A tank initially contains 1000 gallons of water, in which is dissolved 20 pounds of salt. A valve is opened and water containing 0.2 pounds of salt per gallon flows into the tank at a rate of 5 gal/min. The resulting mixture, which is assumed to be always well stirred, drains from the tank at a rate of 5 gal/min.

(a) Write down a differential equation for $S(t)$, the amount of salt in the tank after t minutes. Be sure to state your initial condition, including the units involved.

The initial condition is: $\boxed{S(0) = 20 \text{ lb}}$. Meanwhile, we note that since $S(t)$ is measured in lbs, we have that $S'(t)$ = net rate of chg. of amt of salt, in lb/min; furthermore, at any time t, the current concentration of salt in the tank is $S(t)/1000$, measured in lb/gal.

We have that $S'(t)$ = net rate of chg of salt = $\left(\text{Salt rate in}\right) - \left(\text{Salt rate out}\right) = \left(\text{incoming salt concentration}\right)\left(\text{pumping rate in}\right) - \left(\text{outgoing concentration}\right)\left(\text{pump rate out}\right)$

$= \left(0.2 \frac{\text{lb}}{\text{gal}}\right)\left(5 \frac{\text{gal}}{\text{min}}\right) - \left(\frac{S(t)}{1000} \frac{\text{lb}}{\text{gal}}\right)\left(5 \frac{\text{gal}}{\text{min}}\right)$, since we're assuming the outgoing concentration equals the current concentration. Thus, $\boxed{S'(t) = 1 - \frac{S(t)}{200}}$.

(b) By solving the differential equation, find the amount of salt in the tank after 60 minutes.

$$\frac{dS}{dt} = 1 - \frac{S}{200} = -\frac{1}{200}(S - 200).$$ We can separate variables and integrate:

$$\Rightarrow \int \frac{dS}{S-200} = \int -\frac{1}{200}\, dt = -\frac{1}{200}\int dt$$

$$\Rightarrow \ln|S-200| = -\frac{1}{200}t + C. \quad (\text{any } C)$$

Thus, $|S-200| = e^{-\frac{1}{200}t + C}$, so $S - 200 = \pm e^{C - \frac{1}{200}t} = \pm e^{C} e^{-\frac{1}{200}t}$

$= Ae^{-\frac{1}{200}t}$ (any $A \neq 0$).

But since $S = 20$ when $t = 0$, we have $20 - 200 = Ae^{0} = A$, so $A = -180$.

Thus, $S(t) = 200 + (-180)e^{-\frac{1}{200}t}$,

and so $S(60) = 200 - 180e^{-\frac{60}{200}} = \boxed{200 - 180e^{-3/10} \text{ pounds}}$.

8. (8 points)

(a) Solve the initial value problem

$$\frac{dy}{dx} = x^3y^2, \quad y(0) = -1.$$

Equation is separable, so $\quad \dfrac{dy}{y^2} = x^3 dx$

$$\Rightarrow \int \frac{dy}{y^2} = \int x^3 dx$$

$$\Rightarrow -\frac{1}{y} = \frac{x^4}{4} + C.$$

Since $y=-1$ when $x=0$, we have $-\frac{1}{-1} = 0 + C \Rightarrow C = 1$.

Thus, we solve for y in terms of x to obtain $y = \dfrac{-1}{x^4/4+1}$,

i.e. $\boxed{y = \dfrac{-4}{x^4+4}}$.

(b) Is there a function $y(x)$ satisfying the above differential equation, ~~with~~ [but instead with different initial value] $y(0) = 0$? Explain.

(Note: clearly you can't have $y(x)$ satisfying both $y(0)=1$ and $y(0)=0$; the question is whether the initial value problem $\frac{dy}{dx} = x^3y^2$, $y(0)=0$, has a solution.)

If $-\frac{1}{y} = \frac{x^4}{4} + C$, i.e. if $y = \dfrac{-1}{x^4/4 + C}$, there is no value of C for which we can have $y=0$ when $x=0$. However, recall that separation of variables can omit solutions where $\frac{dy}{dx}$ is the function 0, i.e. where $y = \text{const.}$ (otherwise known as equilibrium solutions). In fact if $\frac{dy}{dx}=0$ for all x, then $x^3y^2=0$ for all x, and so we could have $y=0$ (the constant function 0). Notice that $y(x)=0$ also satisfies $y(0)=0$. Thus, $\boxed{y=0}$ is the function we're looking for.

9. (18 points) At the start of a late-night study session in your dorm, your RA puts out a large bowl of ChexMix®, containing 900 pieces of the delicious snack. Let $y = y(t)$ stand for the amount of ChexMix in the bowl, in *hundreds* of pieces, after t hours.

(a) The more that's in the bowl, the more people are inclined to take out a snack. Suppose that one-third of the pieces in the bowl are removed each hour. Write a *differential equation* satisfied by y in this case, including the initial value of y.

$$y' = -\frac{y}{3},$$

$$y(0) = 9 \quad \text{(hundreds of pcs.)}$$

(b) Find an expression for $y(t)$ in the above situation.

$y(t)$ satisfies natural decay, so we can write $y = y(0) \cdot e^{-\frac{1}{3}t}$, i.e. $\boxed{y(t) = 9e^{-t/3}}$.

Aside/subtle point: Some might interpret the statement of part (a) more literally, employing a kind of "one-third life" (think "half-life") idea: that is, if $y(0)=9$, then $y(1) = 9-\frac{9}{3} = 6$, and $y(2) = 6-\frac{6}{3} = 4$, and $y(3) = 4-\frac{4}{3}$, etc. In this case, the decay is still exponential but actually satisfies $y(t) = 9\cdot\left(\frac{2}{3}\right)^t = 9e^{t\cdot \ln(2/3)}$ instead. For such a function, the diff. eqn. in part (a) would still be $\frac{dy}{dt} = ky$, but where $k = \ln(2/3)$. We should view the above equation, where $k = -1/3$, as an approximation of the literal interpretation, since $\ln(2/3) = \ln(1-1/3) \approx -1/3$. Either interpretation is acceptable, so long as you keep consistent with your choice (and thus with your value of k)!

(c) For the rest of this problem, suppose that your RA is also continually re-supplying the bowl, adding ChexMix at a rate of $\frac{12}{y}$ hundred pieces per hour. Write a new differential equation satisfied by y in this case.

$$y' = -\frac{y}{3} + \frac{12}{y}$$

(d) Find the equilibrium amount of ChexMix in this situation.

If $\left\{ \begin{array}{l} y = \text{constant} \\ \frac{dy}{dt} = 0 \end{array} \right\}$, then $0 = -\frac{y}{3} + \frac{12}{y} \Rightarrow y^2 = 36$,

so that $y = 6$ or $y = -6$. Only $\boxed{y = 6 \text{ (hundred pcs)}}$ makes sense for this model, since you can't have a negative number of pieces.

(e) Use Euler's method with $h = 2$ to estimate the amount of ChexMix left after 4 hours.

We'll need $\frac{4}{2} = 2$ steps to estimate $y(4)$. We have $(t_0, y_0) = (0, 9)$.

Thus, $t_1 = t_0 + h = 0 + 2 = 2$ (hrs) and

$$y_1 = y_0 + h \cdot \left(-\frac{9}{3} + \frac{12}{9}\right) = 9 + 2 \cdot \left(-\frac{9}{3} + \frac{12}{9}\right) = 9 - 6 + \frac{24}{9} = 3 + \frac{8}{3} = \frac{17}{3} \text{ (hundred pcs)},$$

So $t_2 = t_1 + h = 2 + 2 = 4$ (hrs) and

$$y_2 = y_1 + h \cdot \left(-\frac{17/3}{3} + \frac{12}{17/3}\right) = \frac{17}{3} + 2\left(-\frac{17}{9} + \frac{36}{17}\right).$$

Thus, $y(4) \approx y_2 = \boxed{\frac{17}{3} + 2\left(-\frac{17}{9} + \frac{36}{17}\right)}$ hundred pcs.

(f) Find an exact expression for $y(t)$ in this situation.

We can separate variables and integrate: $\frac{dy}{dt} = \frac{1}{3}\left(-y + \frac{36}{y}\right) = \frac{1}{3}\left(\frac{36 - y^2}{y}\right)$,

so $\int \frac{y\,dy}{36 - y^2} = \int \frac{dt}{3}$. The left-hand integral evaluates to $-\frac{1}{2}\ln|36 - y^2|$ (via $u = 36 - y^2$; $du = -2y\,dy$ etc.),

so $-\frac{1}{2}\ln|36 - y^2| = \frac{t}{3} + C$ (any C)

$\Rightarrow \ln|36 - y^2| = -\frac{2t}{3} - 2C$

$\Rightarrow |36 - y^2| = e^{-2t/3 - 2C} \Rightarrow 36 - y^2 = \pm e^{-2C} e^{-2t/3} = Ae^{-2t/3}$ (any $A \neq 0$);

it follows that $y = \pm\sqrt{36 - Ae^{-2t/3}}$. But since $y = 9$ when $t = 0$, we may discard the negative square root and find that $36 - Ae^0 = 81$; i.e. $A = -45$. Thus, $\boxed{y(t) = \sqrt{36 + 45e^{-2t/3}}}$.

10. (16 points) In a certain closed ecosystem, let functions $x(t)$ and $y(t)$ represent the population sizes (in thousands of beings) of two species, X and Y, respectively; here the time t is measured in months. Suppose further that the population sizes are modeled by the equations

$$\frac{dx}{dt} = x - \frac{x^2}{4} - \frac{xy}{4}$$
$$\frac{dy}{dt} = -\frac{y}{4} + \frac{xy}{4}$$

(a) This system is a predator-prey model. Explain why, and determine which species is predator and which is prey.

The "xy" term in each growth rate tells us the effect that an interaction between X & Y has on the growth rate of each species; we see that X's growth rate is diminished by its interactions with Y, while the growth rate of Y is augmented by interactions with X (check out the signs). This is consistent with Y being the predator and X being the prey.

(b) Find all equilibrium solutions to this system.

Need both $\frac{dx}{dt}=0$ and $\frac{dy}{dt}=0$, i.e. constant x and constant y. Note that

$\frac{dx}{dt}=0 \Rightarrow x(1-\frac{x}{4}-\frac{y}{4})=0 \Rightarrow x=0$ or $x+y=4$; and

$\frac{dy}{dt}=0 \Rightarrow y(-\frac{1}{4}+\frac{x}{4})=0 \Rightarrow y=0$ or $x=1$. Based on this second equation, note that if $y=0$, then the first equation is satisfied for either $x=0$ or $x+0=4$. However, if $y\neq 0$, then we must have $x=1$, and then the first equation is satisfied only if $1+y=4$. Thus, the equilibria are $\boxed{(x,y)=(0,0) \text{ or } (4,0) \text{ or } (1,3)}$.

(c) Suppose that at time $t = 0$ months, we have $x(0) = 3$ and $y(0) = 0$. (Thus, there are no beings of species y at any time.) Solve for an explicit formula that gives the population size $x(t)$ in terms of t; what happens to x as t approaches infinity?

(Notice $y=0 \Rightarrow \frac{dy}{dt}=0$ always.) We find $\frac{dx}{dt}=x-\frac{x^2}{4}$, since $y=0$ always.

We can rewrite as $\frac{dx}{dt}=x(1-\frac{x}{4})$; this is a logistic equation with $k=1$ and $K=4$.

Thus, either $x(t)=0$ (which isn't true here) or $x(t)=\frac{4}{1+Ae^{-t}}$ for some A.

But $x=3$ when $t=0$, so $3=\frac{4}{1+Ae^0} \Rightarrow 1+A=\frac{4}{3} \Rightarrow A=\frac{1}{3}$,

So $\boxed{x(t)=\frac{4}{1+\frac{1}{3}e^{-t}}}$. In the "long run," $\lim_{t\to\infty} x(t)=\lim_{t\to\infty}\frac{4}{1+\frac{1}{3}e^{-t}}=\frac{4}{1+0}=\boxed{4 \text{ thousand beings}}$.

For quick reference, here again is the system:

$$\frac{dx}{dt} = x - \frac{x^2}{4} - \frac{xy}{4} = \frac{x}{4}(4-x-y)$$
$$\frac{dy}{dt} = -\frac{y}{4} + \frac{xy}{4} = \frac{y}{4}(x-1)$$

(d) Suppose instead that at time $t = 0$ months, we have $x(0) = 3$ and $y(0) = 4$. Use the differential equations to predict the sizes of the two populations in one month's time; be as mathematically precise as possible.

At time $t=0$, we have $\frac{dx}{dt} = 3 - \frac{9}{4} - \frac{12}{4} = -\frac{9}{4}$ (thous. beings/month) (so X is decreasing in size),

and $\frac{dy}{dt} = -\frac{4}{4} + \frac{12}{4} = 2$ (thous. beings/month) (so Y is increasing in size).

Thus, by a simple-minded linear approximation (i.e. Euler's method with stepsize $h=1$ month)

we could guess that at time $t=1$, $x(1) \approx x(0) + \left(-\frac{9}{4}\ \frac{\text{thous. beings}}{\text{mo}}\right)(1\text{ mo}) = 3 - \frac{9}{4} = \frac{3}{4}$ thous. beings,

and $y(1) \approx y(0) + \left(2\ \frac{\text{thous. beings}}{\text{mo}}\right)(1\text{ mo}) = 4+2 = 6$ thous. beings.

(We could try to be more precise with a smaller step size, but the above is sufficient as a first prediction!)

(e) For the initial conditions of part (d), consider the *signs* of dx/dt and dy/dt at $t = 0$. Based the prediction you made in part (d), make a further prediction about whether dx/dt or dy/dt will change sign at some point *after* the first month. Explain fully how you are able to tell.

- At $t=0$, we had $\frac{dx}{dt} < 0$ (i.e. x decreasing) and $\frac{dy}{dt} > 0$ (i.e. y increasing).
- Notice that if x & y are assumed positive, then $\frac{dx}{dt} = \frac{x}{4}(4-x-y)$ is negative if and only if $x+y>4$,

  and $\frac{dy}{dt} = \frac{y}{4}(x-1)$ is positive if and only if $x>1$.

Thus, based on our prediction that $x(1) \approx \frac{3}{4} < 1$, we see that $\frac{dy}{dt}$ will now be <u>negative</u> (i.e. y decreasing; intuitively the prey population has shrunk to such an extent that the predators cannot survive in large numbers). So a sign change in dy/dt <u>does</u> occur at some point — perhaps during the first or second month, depending on the prediction for $x(1)$. It is reasonable to make the further guess that as y decreases further, ultimately $x+y$ will decrease to below 4, at which point $\frac{dx}{dt}$ will also change sign, from negative to positive (intuitively, the prey has a comeback when the predators are few enough).

11. (10 points) In each of the problems below, indicate clearly what facts you use and how you apply them.

(a) Determine whether $\sum_{n=2}^{\infty} \frac{2}{n^2-1}$ converges or diverges; also, *find the sum* if it converges.

We could show that this series converges using an argument based on the Limit Comparison Test (with $a_n = \frac{2}{n^2-1}$ and $b_n = \frac{1}{n^2}$; here clearly $\sum b_n$ is convergent because it's a p-series with $p=2>1$, etc.); <u>however</u>, that won't help us to find its sum!

Thus, we recall that for non-geometric series, the simplest type of convergent series whose sum we can calculate are the <u>telescoping series</u>. To see if $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$ telescopes, we'll use partial fraction decomposition to write

$$\frac{2}{n^2-1} = \frac{2}{(n+1)(n-1)} = \frac{A}{n-1} + \frac{B}{n+1} \; ; \quad \text{so} \quad 2 = A(n+1)+B(n-1)$$
$$= (A+B)n + (A-B).$$

Thus, $A+B=0$ and $A-B=2$, so we find that $A=1$ and $B=-1$. Next, we'll write a few partial sums of $\sum_{n=2}^{\infty}\frac{2}{n^2-1}$, starting with $\sum_{n=2}^{4}\frac{2}{n^2-1} = \sum_{n=2}^{4}\left(\frac{1}{n-1}-\frac{1}{n+1}\right)$:

$$\sum_{n=2}^{4}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \left(1-\cancel{\tfrac{1}{3}}\right)+\left(\tfrac{1}{2}-\tfrac{1}{4}\right)+\left(\cancel{\tfrac{1}{3}}-\tfrac{1}{5}\right) = 1+\tfrac{1}{2}-\left(\tfrac{1}{4}+\tfrac{1}{5}\right),$$

$$\sum_{n=2}^{5}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \left(1-\cancel{\tfrac{1}{3}}\right)+\left(\tfrac{1}{2}-\cancel{\tfrac{1}{4}}\right)+\left(\cancel{\tfrac{1}{3}}-\tfrac{1}{5}\right)+\left(\cancel{\tfrac{1}{4}}-\tfrac{1}{6}\right) = 1+\tfrac{1}{2}-\left(\tfrac{1}{5}+\tfrac{1}{6}\right),$$

$$\sum_{n=2}^{6}\left(\frac{1}{n-1}-\frac{1}{n+1}\right) = \sum_{n=2}^{5}\left(\frac{1}{n-1}+\frac{1}{n+1}\right)+\left(\tfrac{1}{5}-\tfrac{1}{7}\right) = 1+\tfrac{1}{2}-\cancel{\tfrac{1}{5}}-\tfrac{1}{6}+\cancel{\tfrac{1}{5}}-\tfrac{1}{7} = 1+\tfrac{1}{2}-\left(\tfrac{1}{6}+\tfrac{1}{7}\right);$$

So we'll conclude that the partial sum that goes from $n=2$ to $n=k$ looks like

$$\sum_{n=2}^{k}\left(\frac{1}{n-1}+\frac{1}{n+1}\right) = 1+\frac{1}{2}-\left(\frac{1}{k}+\frac{1}{k+1}\right).$$

Thus, $$\sum_{n=2}^{\infty}\frac{2}{n^2-1} = \lim_{k\to\infty}\left[\sum_{n=2}^{k}\left(\frac{1}{n-1}+\frac{1}{n+1}\right)\right] = \lim_{k\to\infty}\left(1+\frac{1}{2}-\frac{1}{k}-\frac{1}{k+1}\right) = 1+\frac{1}{2}-0-0 = \boxed{\frac{3}{2}};$$

implicity, the series converges since its partial sums have limit $3/2$ (and this is the sum).

(b) Determine whether $\sum_{n=0}^{\infty} \frac{6^n}{5^n + 6^n}$ converges or diverges.

Let $a_n = \frac{6^n}{5^n+6^n}$. Then
$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{6^n}{5^n+6^n}$$
$$= \lim_{n\to\infty} \frac{6^n}{(5^n+6^n)} \cdot \frac{1/6^n}{1/6^n}$$
$$= \lim_{n\to\infty} \frac{1}{5^n/6^n + 1}$$
$$= \lim_{n\to\infty} \frac{1}{(5/6)^n + 1} = \frac{1}{0+1} = 1 \neq 0$$

(note $0 < 5/6 < 1$, so that $\lim_{n\to\infty} (5/6)^n = 0$); thus, the series diverges by the Test for Divergence.

(Note: if you try using the Ratio Test, you get an inconclusive result, because if $a_n = \frac{6^n}{5^n+6^n} = \frac{1}{(5/6)^n+1}$ as above, then
$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(5/6)^n + 1}{(5/6)^{n+1}+1} = \frac{0+1}{0+1} = 1 \ !$$
So you get nowhere.)

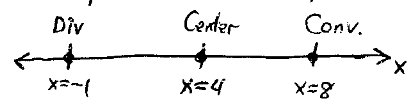
12. (5 points) Suppose we know that the power series

$$\sum_{n=0}^{\infty} c_n(x-4)^n$$

converges if $x = 8$ and diverges if $x = -1$. We are given no other information about this series. For each of the following statements, circle

(Interval of convergence includes 8 but not -1)

- **T** if the statement must be true,
- **F** if the statement must be false, and
- **X** if the statement could be either true or false.



You do not need to justify your answers.

Center: $a=4$

T **F** **X** If R is the radius of convergence of the series, then $4 \le R \le 5$.

Convergence at $x=8$ tells us $R \geq |8-a| = 4$; divergence at $x=-1$ tells us $R \leq |(-1)-a| = 5$.

T **F** **X** The series converges for $x = 0$.

If $R=4$, we could have either convergence or divergence at $x = a-R = 4-4 = 0$.

T **F** **X** The series diverges for $x = 9$.

If $R=5$, we could have either convergence or divergence at $x = a+R = 4+5 = 9$.

T **F** **X** The series diverges for $x > 9$.

If $x > 9$, then $|x-a| = |x-4| = x-4 > 5$; since $R \leq 5$ we have divergence for $|x-a| > 5 \geq R$.

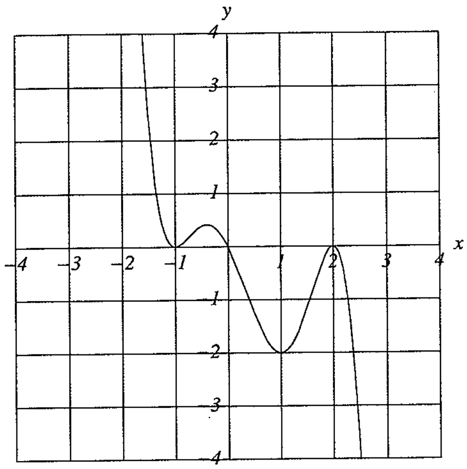
T **F** **X** $\lim_{n\to\infty} c_n = 0$.

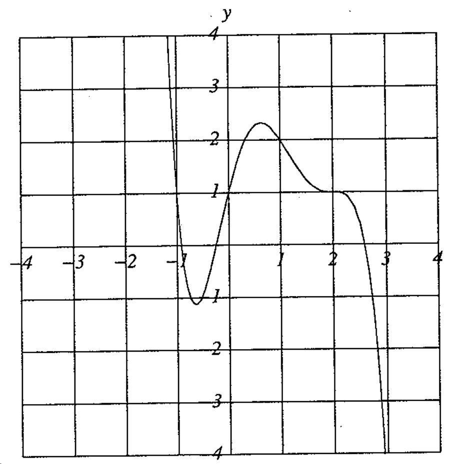
Let $x=5$; since $|x-a| = 5-4 = 1 < 4 \leq R$, have convergence for $x=5$;

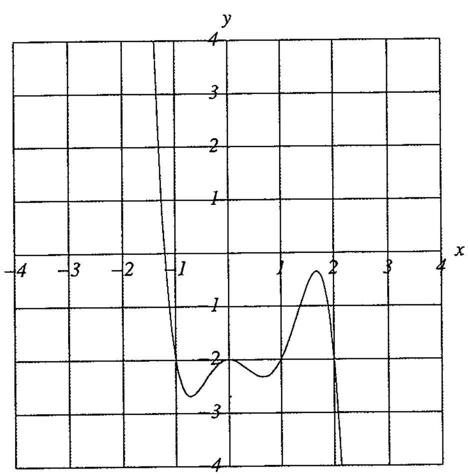
thus, $\sum_{n=0}^{\infty} c_n(5-4)^n = \sum_{n=0}^{\infty} c_n \cdot 1^n = \sum_{n=0}^{\infty} c_n$ converges;

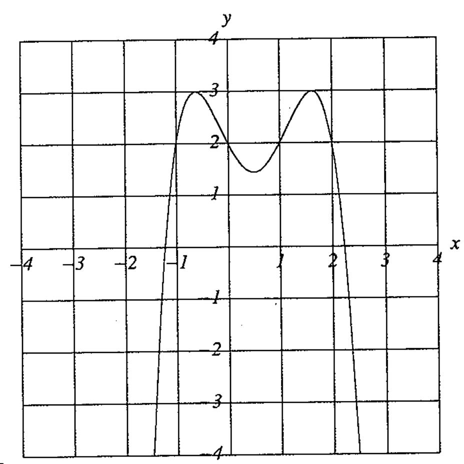
thus, $\lim_{n\to\infty} c_n = 0$.

13. (8 points) The polynomials in the chart below are second-degree Taylor polynomials for functions whose graphs are given below. Match each Taylor polynomial with the appropriate graph.


I


II


III


IV

Key: Center is $a=1$; then $T_2(x) = f(1) + f'(1)\cdot(x-1) + \frac{f''(1)}{2}\cdot(x-1)^2$. Compare coeffs below to values of fns, slopes of fns, & concavities of fns at $x=1$.

Taylor polynomial	Function (I, II, III or IV)
$T_2(x) = 2 + 2(x-1) + (x-1)^2$	IV
$T_2(x) = -2 + 2(x-1) + 3(x-1)^2$	III
$T_2(x) = -2 + \frac{7}{2}(x-1)^2$	I
$T_2(x) = 2 - \frac{3}{2}(x-1) - (x-1)^2$	II

14. (13 points)

(a) Find, showing all your steps, the degree-6 Taylor polynomial $T_6(x)$ with center 0 for the function

$$f(x) = e^x + e^{-x}.$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$C_n = \frac{f^{(n)}(0)}{n!}$
0	$e^x + e^{-x}$	$1+1=2$	$2/0! = 2$
1	$e^x - e^{-x}$	$1-1=0$	0
2	$e^x + e^{-x}$	2	$2/2! = 1$
3	$e^x - e^{-x}$	0	0
4	$e^x + e^{-x}$	2	$2/4!$
5	$e^x - e^{-x}$	0	0
6	$e^x + e^{-x}$	2	$2/6!$

Thus, $T_6(x) = \sum_{n=0}^{6} C_n(x-0)^n = \boxed{2 + x^2 + \frac{2}{4!}x^4 + \frac{2}{6!}x^6}$.

(Alternate method: compute the Taylor poly for e^x to be $1 + \frac{x}{1!} + \cdots + \frac{x^6}{6!}$; now substitute $-x$ into the expression to get the Taylor poly for e^{-x}; then add these together.)

(b) Use T_6 to obtain an estimate for $\sqrt{e} + \frac{1}{\sqrt{e}}$. (You do not need to simplify your answer.)

$$\sqrt{e} + \frac{1}{\sqrt{e}} = e^{1/2} + e^{-1/2} = f\left(\tfrac{1}{2}\right).$$

Thus, $f\left(\frac{1}{2}\right) \approx T_6\left(\frac{1}{2}\right) = 2 + \left(\frac{1}{2}\right)^2 + \left(\frac{2}{4!}\right)\left(\frac{1}{2}\right)^4 + \left(\frac{2}{6!}\right)\left(\frac{1}{2^6}\right)$

$$= \boxed{2 + \frac{1}{4} + \frac{1}{4! \cdot 8} + \frac{1}{6! \cdot 32}}.$$

(c) Compute the 7th derivative $f^{(7)}(x)$ of $f(x)$ and explain why

$$|f^{(7)}(x)| \le 4 \quad \text{for all } x \text{ in } \left[-\tfrac{1}{2}, \tfrac{1}{2}\right].$$

We have $f^{(7)}(x) = e^x - e^{-x}$. Thus, for all x,

$$|f^{(7)}(x)| = |e^x - e^{-x}| \le |e^x| + |e^{-x}|$$
$$= e^x + e^{-x} \quad (\text{since } e^x \,\&\, e^{-x} \text{ are positive});$$

we note that on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we have

$e^x \le e^{1/2}$ since e^x is an increasing function, and
$e^{-x} \le e^{1/2}$ since e^{-x} is a decreasing function.

Thus, on this interval, $|f^{(7)}(x)| \le e^x + e^{-x} \le e^{1/2} + e^{1/2} = 2e^{1/2} = 2\sqrt{e} < 2\sqrt{4} = 4$, since clearly $e < 4$.

(We could do better: in fact $|f^{(7)}(x)| < 2$ on this interval, but no matter here!)

(d) Use the fact from part (c) (even if you were unable to verify it) to draw a conclusion, in sentence form, about the accuracy of your estimate from part (b); be as mathematically precise as you can, and cite all of your reasoning.

By Taylor's inequality, for any x in the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we have that

$|R_6(x)| = |f(x) - T_6(x)| \le \frac{M}{7!}|x|^7$, where M is such that $|f^{(7)}| \le M$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

By (c), since in fact $|f^{(7)}| \le 4$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$, we may take $M = 4$, so

that $|\text{error}| = |f(x) - T_6(x)| \le \frac{4}{7!}|x|^7$ for x in $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

In particular, for $x = \frac{1}{2}$, we have that

$$|\text{error}| = \left|f\left(\tfrac{1}{2}\right) - T_6\left(\tfrac{1}{2}\right)\right| \le \frac{4}{7!} \cdot \frac{1}{2^7} = \frac{1}{7! \cdot 2^5};$$

this means that the estimate using $T_6\left(\frac{1}{2}\right)$ differs from the true value of $f\left(\frac{1}{2}\right) = \sqrt{e} + \frac{1}{\sqrt{e}}$

by no more than $\frac{1}{7! \cdot 2^5}$ units.