

# Math 42 - Winter 2007 - Exam 1 Solutions

1. (40 points) Evaluate each of the following integrals, showing all of your reasoning.

(a)  $\int_3^4 \frac{x}{\sqrt{25-x^2}} dx$

No need for trig substitution!

Just let  $\begin{cases} u = 25 - x^2 \\ du = -2x dx \end{cases}$  (So if  $x=3$ , then  $u=16$ ,  
and if  $x=4$ , then  $u=9$ .)

Then  $\int_3^4 \frac{x dx}{\sqrt{25-x^2}} = \int_{16}^9 \frac{-\frac{1}{2} du}{\sqrt{u}} = -\frac{1}{2} \int_{16}^9 u^{-1/2} du$

$= -\frac{1}{2} \cdot 2u^{1/2} \Big|_{16}^9 = -\frac{1}{2} (2 \cdot 3 - 2 \cdot 4) = \boxed{1}$ .

(If you did by trig substitution, you may have obtained  $5\cos(\arcsin\frac{3}{5}) - 5\cos(\arcsin\frac{4}{5})$ , which is also equal to 1.)

(b)  $\int_1^{e^2} \ln z dz$

Integrate by parts:  $u = \ln z$      $du = \frac{dz}{z}$   
 $dv = dz$      $v = z$

Then  $\int_1^{e^2} \ln z dz = \left[ uv \right]_{z=1}^{z=e^2} - \int_{z=1}^{z=e^2} v du$

$= z \ln z \Big|_{z=1}^{z=e^2} - \int_{z=1}^{z=e^2} dz$

$= (z \ln z - z) \Big|_{z=1}^{z=e^2}$

$= e^2 \ln e^2 - e^2 - (\ln 1 - 1) = \boxed{e^2 + 1}$ .

(c)  $\int \cos^3 2t dt$       First let  $u=2t$ , so  
 $du=2dt$ .

Now using the Pythagorean identity, we get

$$\begin{aligned} \int \cos^3 2t dt &= \frac{1}{2} \int \cos^3 u du = \frac{1}{2} \int \cos u \cdot \cos^2 u du \\ &= \frac{1}{2} \int \cos u \cdot (1 - \sin^2 u) du \\ &= \frac{1}{2} \int \cos u du - \frac{1}{2} \int \cos u \sin^2 u du \quad \left. \begin{array}{l} \text{using } \begin{cases} w = \sin u \\ dw = \cos u du \end{cases} \end{array} \right\} \\ &= \frac{1}{2} \sin u - \frac{1}{6} \sin^3 u + C \\ &= \boxed{\frac{1}{2} \sin 2t - \frac{1}{6} \sin^3 2t + C} \end{aligned}$$

(d)  $\int z \arctan z dz$

Integrate by parts:  $u = \arctan z$        $du = \frac{dz}{1+z^2}$   
 $dv = z dz$        $v = \frac{1}{2} z^2$

Then  $\int z \arctan z dz = \int u dv = uv - \int v du$   
 $= \frac{1}{2} z^2 \arctan z - \int \frac{1}{2} \cdot \frac{z^2}{1+z^2} dz.$

Now  $\frac{z^2}{1+z^2} = 1 - \frac{1}{1+z^2}$  by long division, so

$$\begin{aligned} \int z \arctan z dz &= \frac{1}{2} z^2 \arctan z - \frac{1}{2} \int \frac{z^2}{1+z^2} dz \\ &= \frac{1}{2} z^2 \arctan z - \frac{1}{2} \int dz + \frac{1}{2} \int \frac{1}{1+z^2} dz \\ &= \boxed{\frac{1}{2} z^2 \arctan z - \frac{1}{2} z + \frac{1}{2} \arctan z + C} \end{aligned}$$

$$(e) \int \sin^2 4x \, dx$$

Using the double-angle identity  $\sin^2 t = \frac{1}{2}(1 - \cos 2t)$ ,

we obtain

$$\begin{aligned} \int \sin^2 4x \, dx &= \int \frac{1}{2}(1 - \cos 8x) \, dx \\ &= \frac{1}{2} \int dx - \frac{1}{2} \int \cos 8x \, dx \end{aligned}$$

$$= \boxed{\frac{x}{2} - \frac{\sin 8x}{16} + C}$$

$$(f) \int (4 - x^2)^{-3/2} \, dx$$

After trying some simpler substitutions that don't work well,

fall back on a trig-based one:

$$\begin{cases} x = 2 \sin \theta \\ dx = 2 \cos \theta \, d\theta \end{cases}$$

$$\text{We get } \int (4 - x^2)^{-3/2} \, dx = \int (4 - 4 \sin^2 \theta)^{-3/2} \cdot 2 \cos \theta \, d\theta$$

$$= \int [4 \cdot (1 - \sin^2 \theta)]^{-3/2} \cdot 2 \cos \theta \, d\theta$$

$$= \int (4 \cos^2 \theta)^{-3/2} \cdot 2 \cos \theta \, d\theta$$

$$= \int \frac{2 \cos \theta \, d\theta}{8 \cos^3 \theta} = \frac{1}{4} \int \sec^2 \theta \, d\theta = \frac{1}{4} \tan \theta + C$$

$$= \boxed{\frac{1}{4} \tan(\arcsin \frac{x}{2}) + C}$$

(Note: this answer is equivalent to  $\frac{x}{4\sqrt{4-x^2}} + C$ .)

2. (10 points) Evaluate the integral  $\int \frac{4x^3}{x^3 - x^2 + 3x - 3} dx = \int \frac{4x^3}{(x-1)(x^2+3)} dx.$

We first perform long division:

$$\begin{array}{r} 4 \\ x^3 - x^2 + 3x - 3 \overline{) 4x^3} \\ \underline{4x^3 - 4x^2 + 12x - 12} \\ 4x^2 - 12x + 12 \end{array}$$

Thus  $\int \frac{4x^3}{x^3 - x^2 + 3x - 3} dx = \int 4 dx + \int \frac{4x^2 - 12x + 12}{x^3 - x^2 + 3x - 3} dx = 4x + \int \frac{4x^2 - 12x + 12}{x^3 - x^2 + 3x - 3} dx.$

We can manage the integrand of the second term by doing partial fraction decomposition:

$$\frac{4x^2 - 12x + 12}{x^3 - x^2 + 3x - 3} = \frac{4x^2 - 12x + 12}{(x-1)(x^2+3)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+3}$$

Multiply by  $(x-1)(x^2+3)$ :  $4x^2 - 12x + 12 = A(x^2+3) + (Bx+C)(x-1).$

If we let  $x=1$ , then  $4-12+12 = A(1+3)$ , so  $4A=4$ , i.e.  $A=1$ .

This simplifies the system to:  $4x^2 - 12x + 12 = x^2 + 3 + (Bx+C)(x-1)$

$$\begin{aligned} \Rightarrow 3x^2 - 12x + 9 &= (Bx+C)(x-1) \\ &= Bx^2 + (C-B)x - C. \end{aligned}$$

Thus,  $B=3$  and  $C=-9$ .

We have  $\int \frac{4x^2 - 12x + 12}{x^3 - x^2 + 3x - 3} dx = \int \frac{1}{x-1} dx + \int \frac{3x-9}{x^2+3} dx = \ln|x-1| + \int \frac{3x}{x^2+3} dx - 9 \int \frac{1}{x^2+3} dx.$

We can accomplish the first integral by  $u$ -substitution:  $\int \frac{3x}{x^2+3} dx = \frac{3}{2} \ln(x^2+3) + C.$

The last integral is done by trig. sub:  $\left\{ \begin{array}{l} x = \sqrt{3} \tan \theta \\ dx = \sqrt{3} \sec^2 \theta d\theta \end{array} \right\} \Rightarrow \int \frac{1}{x^2+3} dx = \int \frac{\sqrt{3} \sec^2 \theta d\theta}{3 \tan^2 \theta + 3} =$

$$= \frac{\sqrt{3}}{3} \int \frac{\sec^2 \theta}{\tan^2 \theta + 1} d\theta = \frac{\sqrt{3}}{3} \int d\theta = \frac{\sqrt{3}}{3} \arctan \frac{x}{\sqrt{3}},$$

so  $\int \frac{4x^3}{(x-1)(x^2+3)} dx = \boxed{4x + \ln|x-1| + \frac{3}{2} \ln(x^2+3) - \frac{9\sqrt{3}}{3} \arctan \frac{x}{\sqrt{3}} + C.}$

3. (6 points) A rocket has vertical position 0 at time 0. The following chart gives the rocket's upward velocity, in meters per second, at time  $t$  seconds.

t	0	2	4	6	8	10	12
v	1	8	25	60	120	200	350

- (a) Using all the data in the chart, write a sum representing an estimate of the rocket's height at time  $t = 12$ , using the *Trapezoidal Rule*. You do not have to simplify or fully evaluate your expression.

The rocket's change in height equals  $\int_0^{12} v(t) dt$ ; since initial height is 0m, this integral also equals the rocket's height at  $t=12$  sec.

With intervals of widths  $\Delta t = 2$  sec, there are  $n=6$  such sub-intervals.

Thus, the trapezoidal approximation is  $T_6$ :

$$\begin{aligned} \text{height} &= \int_0^{12} v(t) dt \approx T_6 = \frac{\Delta t}{2} (v(0) + 2v(2) + 2v(4) + 2v(6) + 2v(8) + 2v(10) + v(12)) \\ &= \boxed{\frac{2}{2} (1 + 2 \cdot 8 + 2 \cdot 25 + 2 \cdot 60 + 2 \cdot 120 + 2 \cdot 200 + 350)} \text{ meters} \end{aligned}$$

- (b) Do the same using *Simpson's Rule*; again, you do not have to simplify the expression.

Since  $n=6$ , which is even, this is a valid value for Simpson's Rule.

The approximation is

$$\begin{aligned} \text{height} &= \int_0^{12} v(t) dt \approx S_6 = \frac{\Delta t}{3} (v(0) + 4v(2) + 2v(4) + 4v(6) + 2v(8) + 4v(10) + v(12)) \\ &= \boxed{\frac{2}{3} (1 + 4 \cdot 8 + 2 \cdot 25 + 4 \cdot 60 + 2 \cdot 120 + 4 \cdot 200 + 350)} \text{ meters} \end{aligned}$$

4. (8 points) Consider the integral  $\int_0^2 \cos(x^2) dx$ .

(a) Estimate the error made in approximating the value of this integral using the Midpoint Rule using  $n = 6$  subintervals. State your answer in a complete sentence.

The Midpoint Rule error estimate involves  $K_2$ , so we need to estimate the size of  $|f''(x)|$  on  $[0, 2]$ , for  $f(x) = \cos(x^2)$ .

Derivatives:  $f'(x) = -2x \sin(x^2)$ ;  $f''(x) = -2 \sin(x^2) + (-2x)(2x)(\cos(x^2))$ ;

so  $|f''(x)| = |-2 \sin(x^2) + 4x^2 \cos(x^2)| \leq 2 \cdot 1 + 4 \cdot 2^2 \cdot 1 = 18$ , due to the fact that  $|\sin(x^2)| \leq 1$  and  $|\cos(x^2)| \leq 1$ , and  $|x^2| \leq 4$  on the interval  $[0, 2]$ .

That is, we may take  $K_2 = 18$ . The Error Bound Formula gives:  $|E_M| \leq \frac{K_2(b-a)^3}{24n^2} = \frac{18 \cdot 2^3}{24 \cdot 6^2} = \frac{1}{6}$ ;

this means that the <sup>absolute value of</sup> error, or difference between actual value of  $\int_0^2 \cos(x^2) dx$  and approximated value, is no more than  $\frac{1}{6}$  units.

(b) Again using the Midpoint Rule, how many subintervals  $n$  would be necessary to guarantee an error of at most  $\frac{1}{1000}$ ? Give a valid  $n$  in simplified form. (As long as you justify your answer, you do not have to worry about finding the best possible value.)

We'll use 18 for  $K_2$ , found in part (a).

We wish to find  $n$  such that  $\frac{K_2 \cdot (b-a)^3}{24n^2} \leq \frac{1}{1000}$ , because in this case, we'd

have  $|E_M| \leq \frac{K_2(b-a)^3}{24n^2} \leq \frac{1}{1000}$ , as desired.

Thus, we must solve  $\frac{K_2 \cdot (b-a)^3}{24n^2} = \frac{18 \cdot 2^3}{24n^2} \leq \frac{1}{1000}$  for  $n$ :

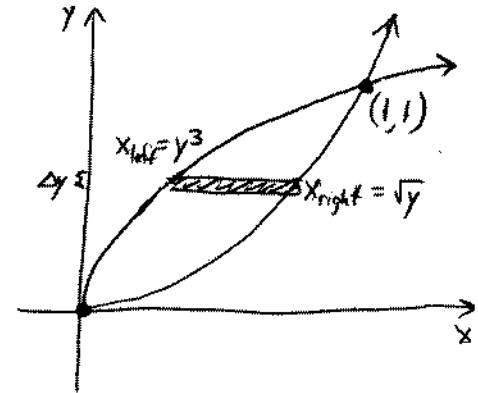
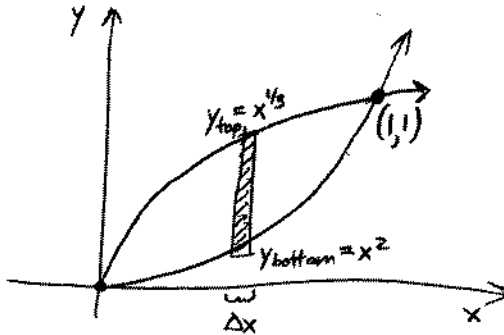
$$\text{So, } 18 \cdot 2^3 \cdot 1000 \leq 24n^2,$$

$$\text{or } n^2 \geq \frac{18 \cdot 8000}{24} = 6000, \text{ i.e. } n \geq \sqrt{6000}.$$

Of course,  $n$  must be a natural number, so  $n = 80$  works (because  $80^2 = 6400 > 6000$ ), and in fact any  $n$  greater than or equal to 78 would be acceptable with this value of  $K_2$ .

5. (10 points)

- (a) Set up two distinct integrals, each in terms of a single variable, representing the area of the region in the first quadrant bounded by the curves  $y = x^2$  and  $y = x^{1/3}$ . For each, make sure you justify your answer (draw a picture and mark a sample slice). Don't evaluate either integral.



Vertical slices: at coordinate  $x$ ,

$$\begin{aligned}\Delta A &= \text{height}(x) \cdot \Delta x \\ &= (y_{\text{top}} - y_{\text{bottom}}) \cdot \Delta x \\ &= (x^{1/3} - x^2) \Delta x,\end{aligned}$$

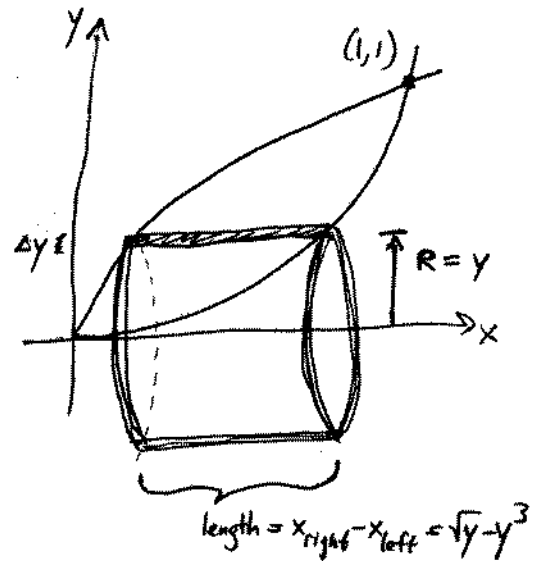
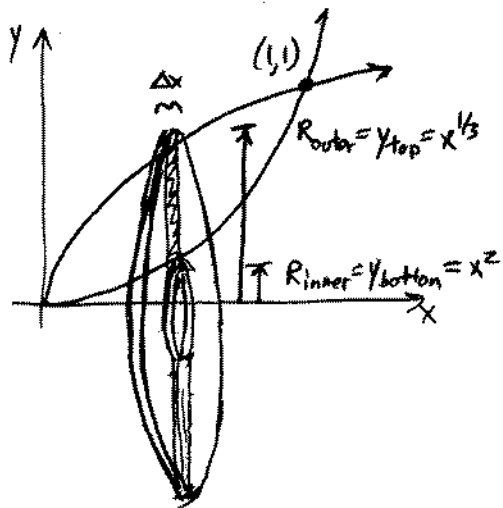
$$\text{so } A = \int_0^1 (x^{1/3} - x^2) dx.$$

Horizontal slices: at coordinate  $y$ ,

$$\begin{aligned}\Delta A &= \text{width}(y) \cdot \Delta y \\ &= (x_{\text{right}} - x_{\text{left}}) \cdot \Delta y \\ &= (\sqrt{y} - y^3) \Delta y,\end{aligned}$$

$$\text{so } A = \int_0^1 (\sqrt{y} - y^3) dy.$$

- (b) Set up two distinct integrals, each in terms of a single variable, representing the volume obtained by rotating the region from part (a) around the  $x$ -axis. For each, make sure you justify your answer (draw a picture, label a sample slice, and cite the method used). Don't evaluate either integral.



Vertical slices  $\Rightarrow$  perp to rot. axis  $\Rightarrow$  Washers.

At coordinate  $x$ ,

$$\begin{aligned}\Delta V &= A(x) \Delta x \\ &= (\pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2) \Delta x \\ &= \pi (x^{2/3} - x^4) \Delta x,\end{aligned}$$

$$\text{so } V = \int_0^1 \pi (x^{2/3} - x^4) dx.$$

Horizontal slices  $\Rightarrow$  parallel to rot. axis  
 $\Rightarrow$  cylindrical shells.

At coordinate  $y$ ,

$$\begin{aligned}\Delta V &= 2\pi(\text{length})(\text{radius})(\text{thickness}) \\ &= 2\pi(\sqrt{y} - y^3)y \Delta y,\end{aligned}$$

$$\text{so } V = \int_0^1 2\pi(\sqrt{y} - y^3)y dy.$$

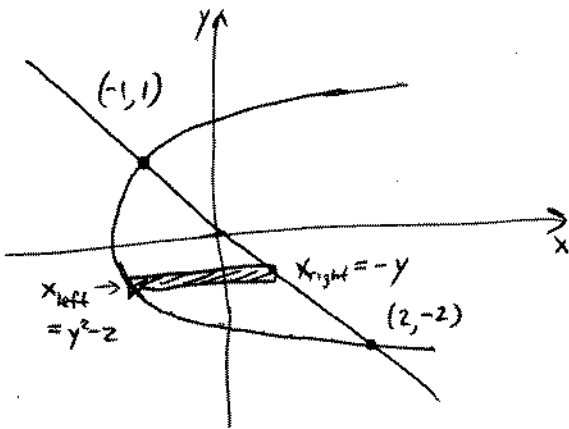
6. (10 points)

- (a) Set up, but do not evaluate, an integral representing the area of the region bounded by the curves  $x = y^2 - 2$  and  $y = -x$ . As justification, draw a picture with a sample slice labeled.

Curves intersect when  $y^2 - 2 = x = -y$ , so  $y^2 + y - 2 = 0$

$$\Rightarrow (y+2)(y-1) = 0, \text{ i.e. } y = -2, 1.$$

When  $y = -2$  then  $x = 2$ ; when  $y = 1$ , then  $x = -1$ . Graphed below:



Much easier to set integral up in terms of  $y$ ;  
i.e., make horizontal slices of thickness  $\Delta y$ .

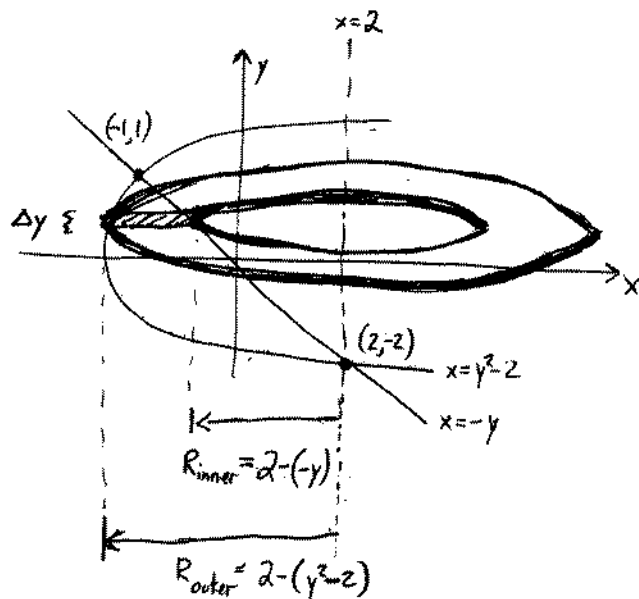
$$\text{Then } \Delta A = \text{length}(y) \cdot \Delta y$$

$$= (x_{\text{right}} - x_{\text{left}}) \Delta y$$

$$= (-y - (y^2 - 2)) \Delta y,$$

$$\text{and so } A = \int_{-2}^1 (-y - y^2 + 2) dy.$$

- (b) Set up an integral representing the volume obtained by rotating the region from part (a) around the line  $x = 2$ . Make sure you justify your answer (draw and label a diagram, and cite the method). Again, don't evaluate the integral.



Horizontal slices  $\Rightarrow$  Slice perp. to rotation axis  $\Rightarrow$  Washers

$$\begin{aligned}
 \text{At coordinate } y, \quad \Delta V &= A(y) \Delta y \\
 &= (\pi R_{\text{outer}}^2 - \pi R_{\text{inner}}^2) \Delta y \\
 &= (\pi(4 - y^2)^2 - \pi(2 + y)^2) \Delta y,
 \end{aligned}$$

$$\text{so } V = \int_{-2}^1 \pi((4 - y^2)^2 - (2 + y)^2) dy.$$

7. (8 points) One very rainy day, a bucket is raised from ground level to the top of a building 200 ft high, using a rope having a linear density of 0.1 lb/ft. Initially (at ground level), the bucket weighs 5 pounds; however, with rain continuing to pour at a constant rate, the bucket takes on water and actually weighs 10 pounds by the moment it reaches the top of the building. Assuming the bucket is being raised at a constant rate, how much work is required to pull the bucket (with its rope) to the top?

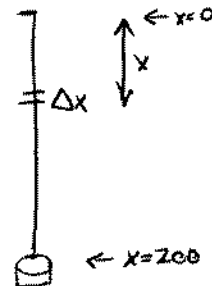
Solution #1: Break into (A) work to lift rope; (B) work to lift bucket and water.

(A) Slice rope into small pieces of length  $\Delta x$  feet;

any such piece  $x$  feet from the top has weight  $(\Delta x)(0.1)$  lb  
and is lifted  $x$  ft,

so work to lift this piece is  $\Delta W = (\text{weight})(\text{dist}) = 0.1x\Delta x$ ,

Meaning total work for rope =  $W = \int_0^{200} 0.1x \Delta x = (0.1) \frac{x^2}{2} \Big|_0^{200} = \underline{2000 \text{ ft}\cdot\text{lb}}$ .



(B) After water has been lifted  $y$  feet, weight of bucket is  $5 + \frac{5}{200}y$  lbs.

(This is because the constant rates imply weight is a linear function of bucket height, and we know weight is 5 when  $y=0$  and weight is 10 when  $y=200$ .)

Thus, process of lifting bucket is the exertion of a variable force  $F(y) = 5 + \frac{5y}{200}$  over

the interval  $0 \leq y \leq 200$ :  $W = \int_0^{200} F(y) dy = \int_0^{200} (5 + \frac{5y}{200}) dy = 5y + \frac{5y^2}{400} \Big|_0^{200} = \underline{1500 \text{ ft}\cdot\text{lb}}$ .

Thus, total work = (A) + (B) =  $2000 + 1500 = \underline{3500 \text{ ft}\cdot\text{lb}}$ .

Solution #2: The whole process of lifting both bucket & rope can be thought of as

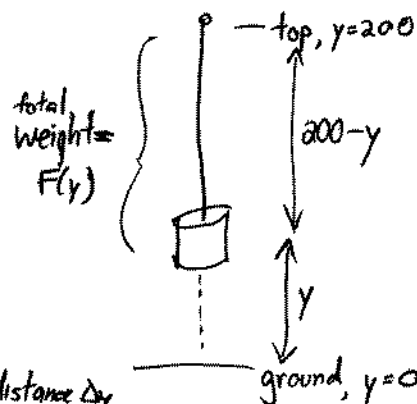
exerting a variable force over a distance.

After the bucket has been hoisted a distance  $y$ , the weight still hanging is (see above discussion):

$$F(y) = \underbrace{5 + \frac{5y}{200}}_{\text{bucket}} + \underbrace{(200-y)(0.1)}_{\text{rope, length times density}}$$

Thus,  $\Delta W = F(y)\Delta y$  is the work to lift this weight a small distance  $\Delta y$ ,

and so  $W = \int_0^{200} F(y) dy = \int_0^{200} (5 + \frac{5y}{200} + (200-y)(0.1)) dy = (\text{etc.}) = \underline{3500 \text{ ft}\cdot\text{lb}}$ , as before.



8. (8 points) A giant clay ant hill in northeastern Argentina has the shape of a perfect cone, with its circular base on the ground. The base has a 1-foot radius, and the ant hill is 3 feet in height. The density of the material is a uniform  $30 \text{ lb/ft}^3$ . How much work have the ants done to assemble this ant hill, lifting all material vertically from ground-level?

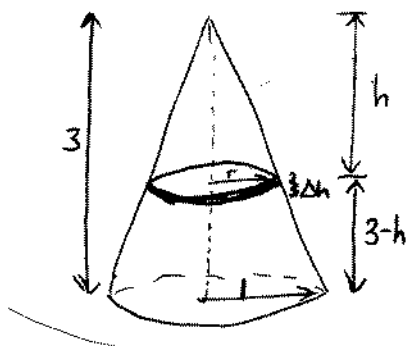


Photo source: Wikipedia

If we slice the cone into horizontal cross-sections of thickness  $\Delta h$  (small), we can use the formula "work = force  $\cdot$  distance" to compute the work to lift each slice, since the force and distance are constant on each slice.

Note that the slice located  $h$  feet from the top of the cone (and therefore  $3-h$  feet from the ground) has radius  $r$ , where  $\frac{r}{h} = \frac{1}{3}$ , i.e.  $r = \frac{h}{3}$ .

Thus,

$$\begin{aligned} \Delta W &= \text{work to lift each slice} = \text{force} \cdot \text{distance} \\ &= (\text{weight of slice})(\text{distance to lift slice}) \\ &= (\text{density})(\text{volume of slice})(\text{distance to lift}) \\ &= (\text{density})(\text{area})(\text{thickness})(\text{dist. to lift}) \\ &= (30)\left(\pi\left(\frac{h}{3}\right)^2\right) \Delta h \cdot (3-h), \end{aligned}$$

$$\begin{aligned} \text{so total work} = W &= \int_{h=0}^{h=3} 30\pi \cdot \frac{h^2}{9} \cdot (3-h) dh = \frac{30\pi}{9} \int_0^3 (3h^2 - h^3) dh = \frac{30\pi}{9} \left( h^3 - \frac{h^4}{4} \right) \Big|_0^3 \\ &= \boxed{\frac{45\pi}{2} \text{ ft}\cdot\text{lb}}. \end{aligned}$$

(There are other ways to set up the integral, but the outcome is the same.)