(1a)

This is an improper integral, since the integrand is undefined at $x=0$.

$$\int_0^1 x^{-1/5}\,dx = \lim_{t\to 0^+} \int_t^1 x^{-1/5}\,dx$$

$$= \lim_{t\to 0^+} \frac{5}{4} x^{4/5}\Big]_t^1 = \frac{5}{4}\lim_{t\to 0^+}\left(1 - t^{4/5}\right)$$

$$= \frac{5}{4}.$$

(1b)

$$\int_0^2 \frac{1}{\sqrt{x}}\,dx$$

$$= \lim_{t\to 0^+} \int_t^2 x^{-1/2}\,dx$$

$$= \lim_{t\to 0^+} \left[2x^{1/2}\right]_t^2$$

$$= \lim_{t\to 0^+} \left[2\sqrt{2} - 2\sqrt{t}\right]$$

$$= \boxed{2\sqrt{2}}$$

(2) If $x \geq 3 > e$, then $\ln x > 1$, so

$\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$, so we check if $\int_3^\infty \frac{1}{\sqrt{x}}\,dx$

diverges.

$$\int_3^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty} \int_3^t x^{-1/2}\,dx = \lim_{t\to\infty} 2x^{\frac{1}{2}}\Big]_3^t$$

$$= 2\lim_{t\to\infty}\left(t^{1/2} - \sqrt{3}\right) = \infty.$$

Since $\int_3^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, and $\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$ for $x \geq 3$, the Comparison Theorem tell us that $\int_3^\infty \frac{\ln x}{\sqrt{x}}\,dx$ diverges also.

(3) Since $|\cos(x)| \le 1$, $\cos^2(x) \le 1$ for all x,

So $\frac{\cos^2 x}{x^3} \le \frac{1}{x^3}$.

Now, $\int_1^\infty \frac{1}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^3}\,dx$

$= \lim_{t\to\infty} \int_1^t x^{-3}\,dx$

$= \lim_{t\to\infty} \left[-\frac{1}{2x^2}\right]_1^t$

$= \lim_{t\to\infty} \left[-\frac{1}{2t^2} + \frac{1}{2}\right]$

$= \frac{1}{2}$.

So $\int_1^\infty \frac{1}{x^3}\,dx$ is convergent, and by the comparison test, this means that

$\int_1^\infty \frac{\cos^2(x)}{x^3}\,dx$ converges.

(5a) True. See Example 4 on p. 426.

(5b) True. The assumption means that the radius of converge of $\sum c_n x^n$ is < 2.

(5c) True.

have $L = 1$.

(6) a) $$\frac{1}{n(n+2)} = \frac{A}{n} + \frac{B}{n+2}$$

$$1 = A(n+2) + Bn$$

$n=0$: $2A = 1$

$n=-2$: $-2B = 1$

$$\frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right].$$

$$S_1 = \frac{1}{2}\left[1 - \frac{1}{3}\right]$$

$$S_2 = \frac{1}{2}\left[1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}\right]$$

$$S_3 = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}\right]$$

$$S_4 = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}\right]$$

$$S_5 = \frac{1}{2}\left[1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{5}} + \cancel{\frac{1}{4}} - \frac{1}{6} + \cancel{\frac{1}{5}} - \frac{1}{7}\right]$$

Now we can start to see the pattern of what's left.

$$S_n = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right].$$

$$\sum_{n=1}^{\infty} \frac{1}{n(n+2)} = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$$= \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4}.$$

b) $$\sum_{k=1}^{\infty} \frac{2^{k-1}}{5^{k+1}} - 3\sum_{k=1}^{\infty} \frac{1}{5^{k+1}}$$

$$= \left(\frac{1}{25} + \frac{2}{5^3} + \frac{4}{5^4} + \dots\right) - 3\left(\frac{1}{25} + \frac{1}{5^3} + \frac{1}{5^4} + \dots\right)$$

$a = \frac{1}{25},\ r = \frac{2}{5}$ $\qquad$ $a = \frac{1}{25},\ r = \frac{1}{5}$

$$= \frac{\frac{1}{25}}{1-\frac{2}{5}} - 3\,\frac{\frac{1}{25}}{1-\frac{1}{5}} = \frac{5}{3}\cdot\frac{1}{25} - 3\cdot\frac{5}{4}\cdot\frac{1}{25}$$

$$= \frac{1}{15} - \frac{3}{20} = -\frac{5}{60} = -\frac{1}{12}.$$

(7) a) $$\lim_{n\to\infty} e^{-\frac{1}{n}} = e^{0} = 1.$$

So $\sum e^{-\frac{1}{n}}$ diverges by the Divergence Test.

b) $\frac{2n}{(n+3)^{3/2}}$ looks similar to $\frac{2n}{n^{3/2}} = \frac{2}{\sqrt{n}}$.

$$\lim_{n\to\infty} \frac{\frac{2n}{(n+3)^{3/2}}}{\frac{2}{\sqrt{n}}} = \lim_{n\to\infty} \frac{n^{3/2}}{(n+3)^{3/2}} = \left(\lim_{n\to\infty} \frac{n}{n+3}\right)^{3/2}$$

$$= 1.$$

So by the Limit Comparison Test, $\sum_{n=1}^{\infty} \frac{2n}{(n+3)^{3/2}}$ does the same thing as $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ is a divergent p-series, the series $\sum_{n=1}^{\infty} \frac{2n}{(n+3)^{3/2}}$ diverges.

c) Let $f(x) = xe^{-x}$.

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} < 0$$

if $x > 1$.

So f is a continuous, decreasing, positive function. By the Integral Test, $\sum_{n=1}^{\infty} ne^{-n}$ does the same thing as $\int_1^{\infty} xe^{-x}\,dx$.

$$\int xe^{-x}\,dx \qquad \begin{array}{ll} u = x & du = dx \\ dv = e^{-x}dx & v = -e^{-x} \end{array}$$

$$= -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C.$$

$$\text{So} \quad \int_1^{\infty} xe^{-x}\,dx = \lim_{t\to\infty} \left[-xe^{-x} - e^{-x}\right]_1^t$$

$$= -\lim_{t\to\infty} \left(\frac{t}{e^t} + \frac{1}{e^t} - 2e^{-1}\right)$$

$$= -\lim_{t\to\infty} \frac{t}{e^t} + 2e^{-1} \qquad \frac{\infty}{\infty} : \text{use l'Hôpital}$$

$$= -\lim_{t\to\infty} \frac{1}{e^t} + 2e^{-1} = 2e^{-1}.$$

Since the integral converges, so does the series.

d) $\frac{3}{k^2+7} < \frac{3}{k^2}$, and $3\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a convergent p-series. So the series converges by the Comparison Test.

8 a) $f(x) = \frac{1}{x \ln x} > 0$ and is clearly decreasing.

$u = \ln x \quad du = \frac{1}{x}dx$

$$\int_2^{\infty} \frac{1}{x\ln x}dx = \int_{\ln 2}^{\infty} \frac{1}{u}du \quad \text{diverges.}$$

So by the Integral Test,

$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges also.

b) Since $\lim_{n\to\infty} \frac{n^3+3n+2}{n^3+6} = 1$, the terms of the series oscillate between being about 1 and about -1. So

$\lim_{n\to\infty} (-1)^n \frac{n^3+3n+2}{n^3+6}$ does not exist,

so by the Divergence Test, $\sum_{n=0}^{\infty} (-1)^n \frac{n^3+3n+2}{n^3+6}$ diverges.

c) <u>Solution I</u>: $\frac{n}{2^n(n+1)} < \frac{n+1}{2^n(n+1)} = \frac{1}{2^n}$

(or $\frac{n}{2^n(n+1)} < \frac{n}{2^n n} = \frac{1}{2^n}$).

Since $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ converges by the Comparison Test.

<u>Solution II</u>: $\lim_{n\to\infty} \frac{\left(\frac{n}{2^n(n+1)}\right)}{\left(\frac{1}{2^n}\right)} = \lim_{n\to\infty} \frac{n}{n+1} = 1 > 0$,

and $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent geometric series, so $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ converges by the Limit Comparison Test.

Solution III:

$$\lim_{n\to\infty} \frac{\left(\frac{n+1}{2^{n+1}(n+2)}\right)}{\left(\frac{n}{2^n(n+1)}\right)} = \lim_{n\to\infty} \frac{2^n}{2^{n+1}} \frac{(n+1)^2}{n(n+2)}$$

$$= \frac{1}{2} \lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} = \frac{1}{2} < 1,$$

So $\sum_{n=1}^{\infty} \frac{n}{2^n(n+1)}$ converges by the Ratio Test.

9) a) $\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{4^n (2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} \left(\frac{\pi}{2}\right)^{2n} = \cos\left(\frac{\pi}{2}\right) = 0.$

b) $\sum_{n=1}^{\infty} \frac{1-2^n}{4^n} = \sum_{n=1}^{\infty} \left[\left(\frac{1}{4}\right)^n - \left(\frac{1}{2}\right)^n\right]$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}\right)^n - \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$$

$a = \frac{1}{4},\ r = \frac{1}{4}$ $\qquad$ $a = \frac{1}{2},\ r = \frac{1}{2}$

$$= \frac{1/4}{1-1/4} - \frac{1/2}{1-1/2}$$

$$= \frac{1}{3} - 1 = -\frac{2}{3}.$$

(10) Begin with the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}}{\sqrt{n+1}}(x+3)^{n+1}}{\frac{(-2)^n}{\sqrt{n}}(x+3)^n}\right| = \lim_{n\to\infty}\left|(-2)(x+3)\frac{\sqrt{n}}{\sqrt{n+1}}\right|$$

$$= 2|x+3|\lim_{n\to\infty}\sqrt{\frac{n}{n+1}} = 2|x+3|.$$

So we get convergence for $2|x+3| < 1$, i.e. if $|x+3| < \frac{1}{2}$, or $-\frac{7}{2} < x < -\frac{5}{2}$.

We still need to check $x = -\frac{7}{2}, -\frac{5}{2}$.

$x = -\frac{7}{2}$: $\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}\left(-\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ diverges

(p-series with $p = \frac{1}{2}$).

$x = -\frac{5}{2}$: $\sum_{n=1}^{\infty}\frac{(-2)^n}{\sqrt{n}}\left(\frac{1}{2}\right)^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$.

Since $\frac{1}{\sqrt{n}}$ decreases to 0, this converges by the Alternating Series Test.

So the interval of convergence is $\left(-\frac{7}{2}, -\frac{5}{2}\right]$.

(11) $\dfrac{x}{2+x} = x \cdot \dfrac{1}{2+x} = x \cdot \dfrac{1}{2(1+\frac{x}{2})}$

$= \dfrac{x}{2} \quad \dfrac{1}{1-(-\frac{x}{2})} = \dfrac{x}{2} \cdot \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n$ for $|-\frac{x}{2}|<1$

or: $|x|<2$

$= \dfrac{x}{2} \sum_{n=0}^{\infty} (-1)^n \dfrac{x^n}{2^n} = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{2^{n+1}} x^{n+1}$

$\uparrow$ this is fine

$= \sum_{n=1}^{\infty} \dfrac{(-1)^{n+1}}{2^n} x^n$

with $R=2$.

(12) a) $\sin^{-1} x + C = \int \frac{1}{\sqrt{1-x^2}} dx$

$= \int \sum_{n=0}^{\infty} \dfrac{(2n)!}{4^n (n!)^2} x^{2n} dx$

$= \sum_{n=0}^{\infty} \dfrac{(2n)!}{4^n (n!)^2} \dfrac{x^{2n+1}}{2n+1}$.

Plug in $x=0$ to find C:

$0 + C = 0$ so $C = 0$.

$\sin^{-1} x = \sum_{n=0}^{\infty} \dfrac{(2n)!}{4^n (2n+1)(n!)^2} x^{2n+1}$ for $-1<x<1$.

b) $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, so

$$\frac{\pi}{6} = \sin^{-1}\left(\frac{1}{2}\right) = \sum_{n=0}^{\infty} \frac{(2n)!}{4^n(2n+1)(n!)^2} \left(\frac{1}{2}\right)^{2n+1}$$

$$= \frac{1}{2} \sum_{n=0}^{\infty} \frac{(2n)!}{16^n (2n+1)(n!)^2}$$

So $$\pi = \sum_{n=0}^{\infty} \frac{3 \cdot (2n)!}{16^n (2n+1)(n!)^2}.$$

13 $$\lim_{x\to 0} \frac{e^{x^2}-1}{\cos x - 1} = \lim_{x\to 0} \frac{\left(1 + x^2 + \frac{1}{2}x^4 + \dots\right) - 1}{\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \dots\right) - 1}$$

$$= \lim_{x\to 0} \frac{x^2 + \frac{1}{2}x^4 + \dots}{-\frac{x^2}{2} + \frac{x^4}{24} + \dots}$$

$$= \lim_{x\to 0} \frac{1 + \frac{1}{2}x^2 + \dots}{-\frac{1}{2} + \frac{x^2}{24} + \dots} = \frac{1}{\left(-\frac{1}{2}\right)} = -2.$$

(14) a) $f(x) = x^{4/3}$ $\qquad f(27) = 3^4 = 81$

$f'(x) = \frac{4}{3}x^{1/3}$ $\qquad f'(27) = \frac{4}{3}\cdot 3 = 4$

$f''(x) = \frac{4}{9}x^{-2/3}$ $\qquad f''(27) = \frac{4}{9}\cdot 3^{-2} = \frac{4}{81}$

$f^{(3)}(x) = -\frac{8}{27}x^{-5/3}$ $\qquad f^{(3)}(27) = -\frac{8}{27}3^{-5} = -\frac{8}{3^8}.$

$f^{(4)}(x) = \frac{40}{81}x^{-8/3}.$

$$T_3(x) = f(27) + f'(27)(x-27) + \frac{f''(27)}{2}(x-27)^2 + \frac{f^{(3)}}{6}(x-27)^3$$

$$= 81 + 4(x-27) + \frac{2}{81}(x-27)^2 - \frac{4}{3^9}(x-27)^3.$$

b) $25 \le x \le 29$ means $|x-27| \le 2.$

Since $f^{(4)}$ is decreasing,

$|f^{(4)}(x)| = \frac{40}{81}x^{-8/3} \le \frac{40}{81}\cdot 25^{-8/3}$ for $|x-27| \le 2.$

So by Taylor's Inequality,

$|R_3(x)| \le \frac{1}{24}\cdot\frac{40}{81}25^{-8/3}\cdot 2^4$, whatever that is.