① a) • Solution 1: $u = r^2 - 4$, $du = 2r\,dr$

$$\int \frac{r}{r^2-4}\,dr = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|r^2-4| + C$$

(Note the absolute values are necessary this time!)

• Solution 2: $\dfrac{r}{r^2-4} = \dfrac{r}{(r+2)(r-2)} = \dfrac{A}{r+2} + \dfrac{B}{r-2}$

$r = A(r-2) + B(r+2)$

Using $r=-2$: $-2 = -4A \Rightarrow A = \frac{1}{2}$

Using $r=2$: $2 = 4B \Rightarrow B = \frac{1}{2}$

So $\dfrac{r}{r^2-4} = \dfrac{1}{2}\left[\dfrac{1}{r+2} + \dfrac{1}{r-2}\right]$ (Check this!)

$$\int \frac{r}{r^2-4}\,dr = \frac{1}{2}\int\left[\frac{1}{r+2} + \frac{1}{r-2}\right]dr$$

$$= \frac{1}{2}\left[\ln|r+2| + \ln|r-2|\right] + C$$

Log properties show this is the same as the first answer.

• Solution 3: $r = 2\sec\theta$, so $r^2 - 4 = 4(\sec^2\theta - 1)$
$= 4\tan^2\theta$,

$dr = 2\sec\theta\tan\theta\,d\theta$

$$\int \frac{r}{r^2-4}\,dr = \int \frac{2\sec\theta}{4\tan^2\theta}\,2\sec\theta\tan\theta\,d\theta$$

$$= \int \frac{\sec^2\theta}{\tan\theta}\,d\theta \qquad u = \tan\theta \quad du = \sec^2\theta\,d\theta$$

$$= \int \frac{1}{u}\,du = \ln|u| + C = \ln|\tan\theta| + C$$

$$= \ln\left|\tan\left(\sec^{-1}\left(\tfrac{r}{2}\right)\right)\right| + C.$$

It's tricky but one can show this is the same as the other answers — except this method technically only works for $r^2 - 4 > 0$ (a point we didn't discuss in this class) since $|\sec\theta| \geq 1$ for any θ, whereas the other solutions are also fine for $r^2 - 4 < 0$.

Compare the three solutions and judge for yourself which are easiest and hardest. Think what this says about which methods to try first.

b) $\dfrac{1}{r^2-4} = \dfrac{A}{r+2} + \dfrac{B}{r-2}$

$1 = A(r-2) + B(r+2)$

Using $r=-2$: $1=-4A \Rightarrow A = -\frac{1}{4}$

Using $r=2$: $1=4B \Rightarrow B=\frac{1}{4}$

$$\int \frac{1}{r^2-4}\,dr = \frac{1}{4}\int \left[\frac{1}{r-2} - \frac{1}{r+2}\right] dr$$

$$= \frac{1}{4}\left[\ln|r-2| - \ln|r+2|\right] + C$$

$$= \frac{1}{4}\ln\left|\frac{r-2}{r+2}\right| + C$$

Either form is fine.

You might also try the trig substitution used in Solution 3 of part a), but that will soon stall at $\int \csc\theta\,d\theta$, which is quite tricky.

c) $\int_1^e x \ln x \, dx$

$u = \ln x \quad dv = x\,dx$

$du = \frac{1}{x}dx \quad v = \frac{1}{2}x^2$

$$= \frac{1}{2}x^2\ln x\Big]_1^e - \int_1^e \frac{1}{2}x^2\cdot\frac{1}{x}\,dx = \frac{1}{2}x^2\ln x\Big]_1^e - \int_1^e \frac{1}{2}x\,dx$$

$$= \left[\frac{1}{2}x^2\ln x - \frac{1}{4}x^2\right]_1^e = \frac{1}{2}e^2 - \frac{1}{4}e^2 - \left(0 - \frac{1}{4}\right)$$

$$= \frac{1}{4}e^2 + \frac{1}{4}.$$

d) $\int \sin^4 x \, dx = \int (\sin^2 x)^2 \, dx$

$$= \int \left[\tfrac{1}{2}(1-\cos 2x)\right]^2 dx$$

$$= \frac{1}{4}\int (1 - 2\cos 2x + \cos^2 2x) \, dx$$

$$= \frac{1}{4}x - \frac{1}{2}\int \cos 2x \, dx + \frac{1}{4}\int \cos^2 2x \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}\int (1+\cos 4x) \, dx$$

$$= \frac{1}{4}x - \frac{1}{4}\sin 2x + \frac{1}{8}x + \frac{1}{32}\sin 4x + C.$$

e) This is an improper integral, since the integrand is undefined at $x=0$.

$$\int_0^1 x^{-1/5} dx = \lim_{t\to 0^+} \int_t^1 x^{-1/5} dx$$

$$= \lim_{t\to 0^+} \frac{5}{4} x^{4/5} \Big]_t^1 = \frac{5}{4} \lim_{t\to 0^+} (1 - t^{4/5})$$

$$= \frac{5}{4}.$$

f) $\int_{\pi/4}^{\pi/2} \sin^2\theta \; \cos\theta \, d\theta$

$u = \sin\theta$
$du = \cos\theta \, d\theta$
when $\theta = \pi/4, u = \frac{1}{\sqrt{2}}$.
when $\theta = \frac{\pi}{2}, u = 1$.

$= \int_{1/\sqrt{2}}^{1} u^2 \, du$

$= \frac{1}{3}u^3\Big]_{1/\sqrt{2}}^{1} = \frac{1}{3}\left(1 - \frac{1}{2^{3/2}}\right)$.

g) If $x = 3\tan\theta$, $x^2+9 = 9(\tan^2\theta + 1)$
$= 9\sec^2\theta$

and $dx = 3\sec^2\theta$, and $\theta = \tan^{-1}\left(\frac{x}{3}\right)$.

$\int \frac{1}{(x^2+9)^{3/2}} dx = \int \frac{3\sec^2\theta}{(9\sec^2\theta)^{3/2}} d\theta = \frac{3}{3^3}\int \frac{\sec^2\theta}{\sec^3\theta} d\theta$

$= \frac{1}{9}\int \frac{1}{\sec\theta} d\theta = \frac{1}{9}\int \cos\theta \, d\theta$

$= \frac{1}{9}\sin\theta + C = \frac{1}{9}\sin\left(\tan^{-1}\left(\frac{x}{3}\right)\right) + C.$

h) $\frac{x^3}{x^3-x^2-x+1}$ does not have lower degree on the top than on the bottom, so we start with long division.

$$
\begin{array}{r|l}
 & \quad 1 \\
\cline{2-2}
x^3-x^2-x+1 & x^3+0x^2+0x-0 \\
 & \underline{x^3-x^2-x+1} \\
 & \qquad x^2+x-1
\end{array}
$$

So $\dfrac{x^3}{x^3-x^2-x+1} = 1 + \dfrac{x^2+x-1}{x^3-x^2-x+1}$.

If you're clever, you might skip the long division just by writing

$$\frac{x^3}{x^3-x^2-x+1} = \frac{x^3-x^2-x+1 \ +x^2+x-1}{x^3-x^2-x+1}$$

$$= 1 + \frac{x^2+x-1}{x^3-x^2-x+1}.$$

Now $\dfrac{x^2+x-1}{x^3-x^2-x+1} = \dfrac{x^2+x-1}{(x+1)(x-1)^2} = \dfrac{A}{x+1} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

$x^2+x-1 = A(x-1)^2 + B(x+1)(x-1) + C(x+1)$.

Using $x=-1$: $-1 = 4A \Rightarrow A = -\frac{1}{4}$.

Using $x=1$: $1 = 2C \Rightarrow C = \frac{1}{2}$.

Using $x=0$: $-1 = A - B + C = -\frac{1}{4} - B + \frac{1}{2}$

$\Rightarrow B = 1 - \frac{1}{4} + \frac{1}{2} = \frac{5}{4}$.

$$\int \frac{x^3}{(x+1)(x-1)^2}\,dx = \int \left[1 + \left(-\tfrac{1}{4}\right)\frac{1}{x+1} + \frac{5}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{(x-1)^2}\right]dx$$

$$= x - \tfrac{1}{4}\ln|x+1| + \tfrac{5}{4}\ln|x-1| - \tfrac{1}{2}\frac{1}{(x-1)} + C.$$

(i) $$\int_1^4 (at+b)\sqrt{t}\,dt = \int_1^4 \left(at^{3/2} + bt^{1/2}\right)dt$$

$$= \left[a\tfrac{2}{5}t^{5/2} + b\tfrac{2}{3}t^{3/2}\right]_1^4$$

$$= a\tfrac{2}{5}2^5 + b\tfrac{2}{3}2^3 - a\tfrac{2}{5} - b\cdot\tfrac{2}{3}$$

$$= \tfrac{2}{5}a(32-1) + \tfrac{2}{3}b(8-1)$$

$$= \frac{62}{5}a + \frac{14}{3}b.$$

② If $x \geq 3 > e$, then $\ln x > 1$, so

$\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$, so we check if $\int_3^\infty \frac{1}{\sqrt{x}}\,dx$

diverges.

$$\int_3^\infty \frac{1}{\sqrt{x}}\,dx = \lim_{t\to\infty}\int_3^t x^{-1/2}\,dx = \lim_{t\to\infty} 2x^{\frac{1}{2}}\Big]_3^t$$

$$= 2\lim_{t\to\infty}\left(t^{1/2} - \sqrt{3}\right) = \infty.$$

Since $\int_3^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, and $\frac{\ln x}{\sqrt{x}} > \frac{1}{\sqrt{x}} > 0$

for $x \geq 3$, the Comparison Theorem tell us

that $\int_3^\infty \frac{\ln x}{\sqrt{x}}\,dx$ diverges also.

③ If $r(t)$ = rate of growth in year t, then the total population growth is $\int_{1980}^{2000} r(t)\,dt$).

Simpson's Rule approximates this by

$$\frac{5}{3}\left[r(1980) + 4r(1985) + 2r(1990) + 4r(1995) + r(2000)\right]$$

$$= \frac{5}{3}\left[1 + (-4) + 0 + 4 + 2\right] = 5.$$

④ a) $f(x) = \sin(x^2)$ $\quad f'(x) = 2x\cos(x^2)$

$f''(x) = 2\cos(x^2) - 4x\sin(x^2)$.

$$|f''(x)| = |2\cos(x^2) - 4x\sin(x^2)|$$

$$\leq 2|\cos(x^2)| + 4|x|\,|\sin(x^2)|$$

$$\leq 2\cdot 1 + 4\cdot 1\cdot 1 = 6$$

using $|\sin\theta| \leq 1$, $|\cos\theta| \leq 1$, and $|x| \leq 1$ for $0 \leq x \leq 1$.

So we can use $K=6$. (A smaller value of K is possible, but it's hard to come by.)

$b-a=1$, and $n=5$, so

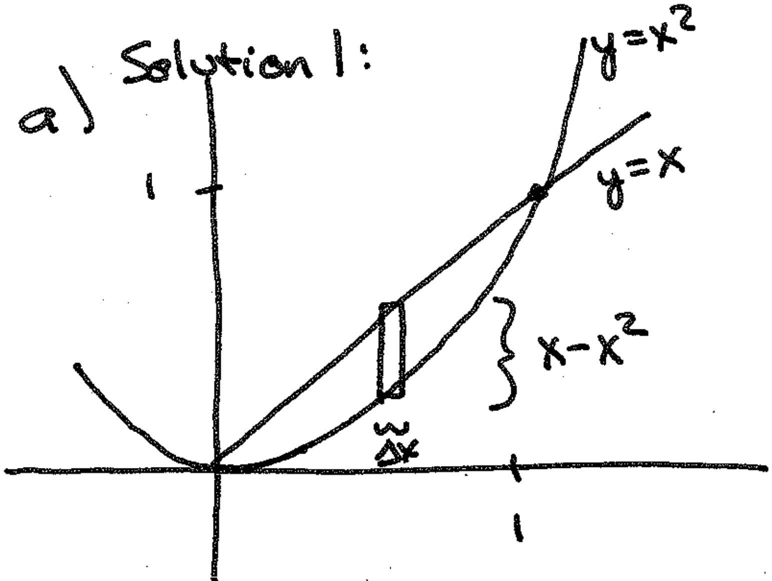
$$|E_T| \le \frac{6\cdot 1^3}{12(5^2)} = \frac{1}{2\cdot 25} = \frac{1}{50}$$

b) Using $K=6$ as above, we want

$$\frac{6\cdot 1^3}{12\,n^2} \le \frac{1}{200}, \text{ so } n^2 \ge 100.$$

So it's good enough to have $n=10$.

5 a) Solution 1:



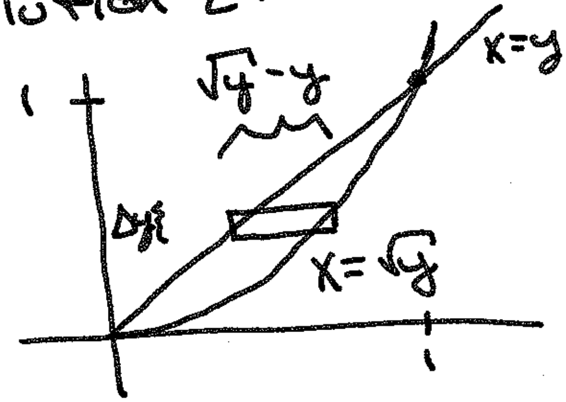
$\Delta A \approx (x-x^2)\Delta x$,

so $A=\int_0^1 (x-x^2)\,dx$

$= \frac{1}{2}x^2-\frac{1}{3}x^3\Big]_0^1$

$= \frac{1}{2}-\frac{1}{3}-0=\frac{1}{6}.$

Solution 2:



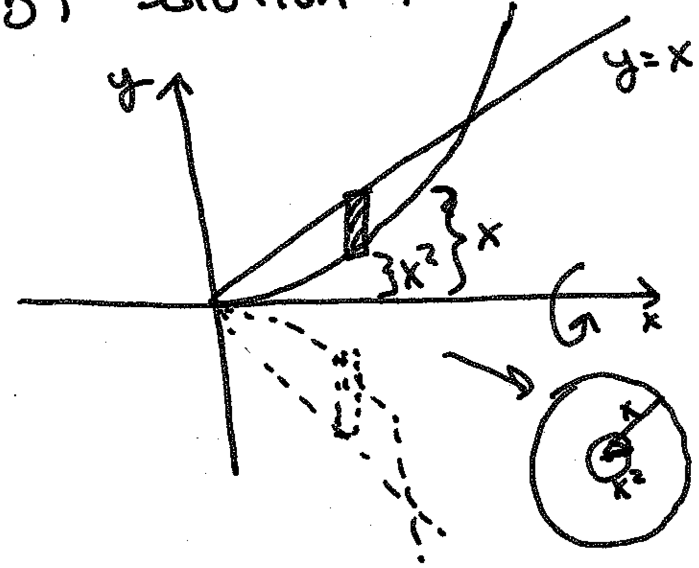
$\Delta A \approx (\sqrt{y}-y)\,\Delta y$

$A=\int_0^1 (y^{1/2}-y)\,dy$

$= \frac{2}{3}y^{3/2}-\frac{1}{2}y^2\Big]_0^1 = \frac{2}{3}-\frac{1}{2}-0$

$= \frac{1}{6}.$

b) Solution 1:



Get fat washers:

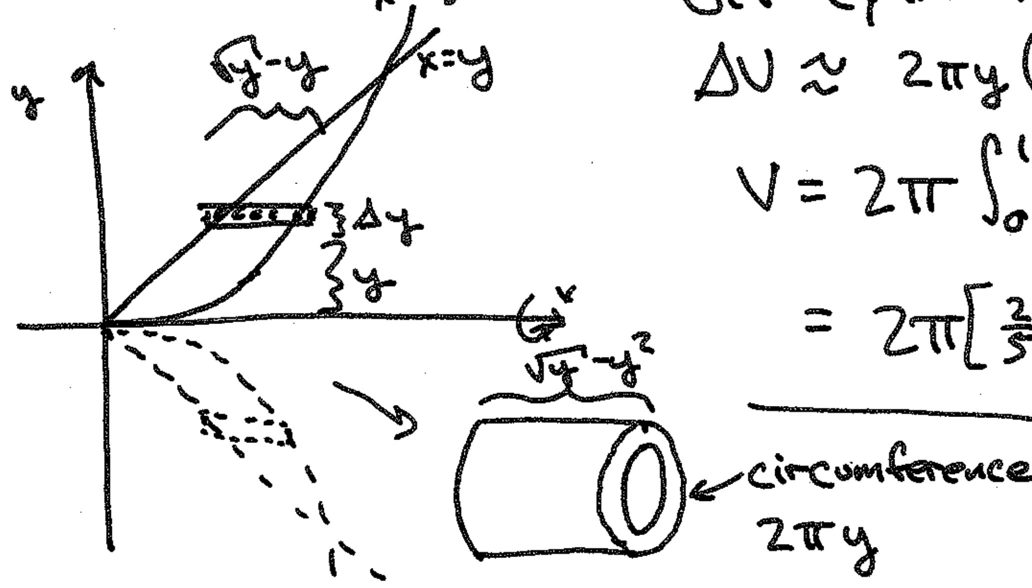
$$\Delta V \approx \left(\pi x^2 - \pi (x^2)^2\right)\Delta x$$

$$V = \pi \int_0^1 (x^2 - x^4)\,dx$$

$$= \pi\left(\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right)\Big]_0^1$$

$$= \pi\left(\tfrac{1}{3} - \tfrac{1}{5}\right) - 0 = \frac{2\pi}{15}.$$

Solution 2:



Get cylindrical shells

$$\Delta V \approx 2\pi y(\sqrt{y} - y)\,\Delta y$$

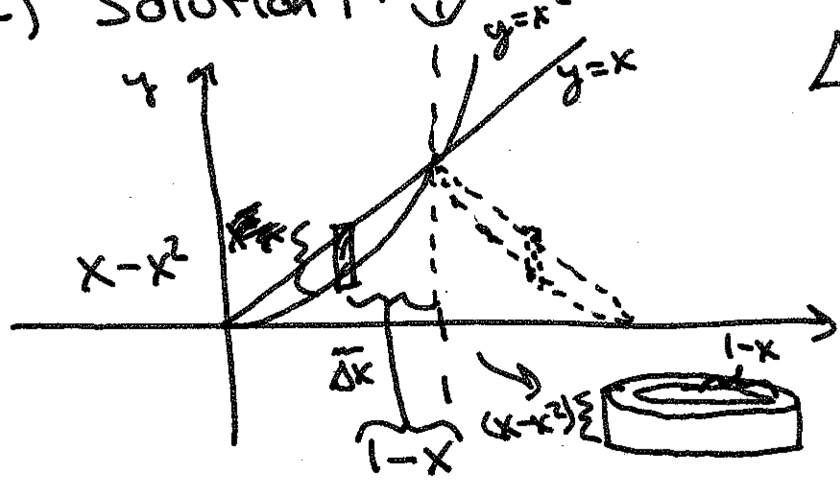
$$V = 2\pi \int_0^1 (y^{3/2} - y^2)\,dy$$

$$= 2\pi\left[\tfrac{2}{5}y^{5/2} - \tfrac{1}{3}y^3\right]_0^1$$

$$= 2\pi\left(\tfrac{2}{5} - \tfrac{1}{3}\right) - 0$$

$$= \frac{2\pi}{15}.$$

c) Solution 1:



$$\Delta V \approx 2\pi(1-x)(x-x^2)\,\Delta x$$

circumference $2\pi(1-x)$.

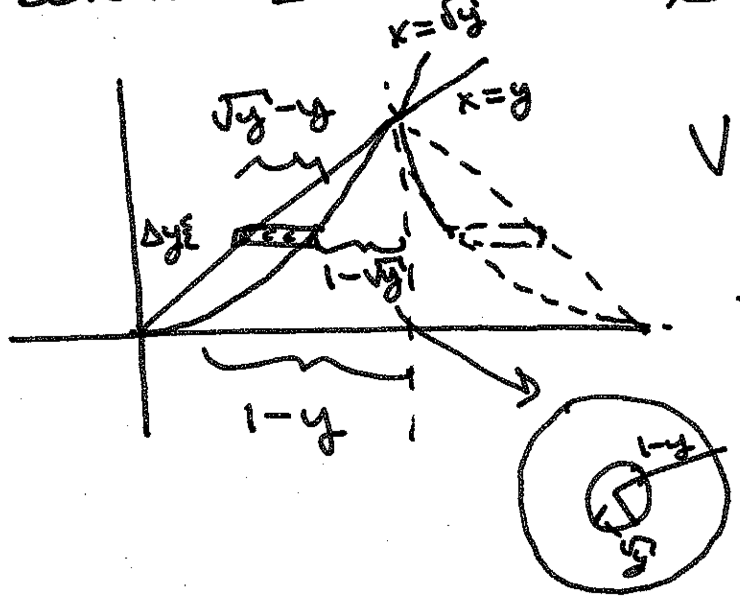
$$V = 2\pi \int_0^1 (1-x)(x-x^2)\,dx \quad \text{(Watch the bounds!)}$$

$$= 2\pi \int_0^1 (x - 2x^2 + x^3)\,dx$$

$$= 2\pi \left[\tfrac{1}{2}x^2 - \tfrac{2}{3}x^3 + \tfrac{1}{4}x^4 \right]_0^1 = 2\pi \left[\tfrac{1}{2} - \tfrac{2}{3} + \tfrac{1}{4} \right] - 0$$

$$= \frac{\pi}{6}.$$

Solution 2:



$$\Delta V \approx \left[\pi(1-y)^2 - \pi(1-\sqrt{y})^2 \right] \Delta y$$

$$V = \pi \int_0^1 \left[1 - 2y + y^2 - (1 - 2\sqrt{y} + y) \right] dy$$

$$= \pi \int_0^1 (-3y + y^2 + 2\sqrt{y})\,dy$$

$$= \pi \left[-\tfrac{3}{2}y^2 + \tfrac{1}{3}y^3 + \tfrac{4}{3}y^{3/2} \right]_0^1$$

$$= \pi \left(-\tfrac{3}{2} + \tfrac{1}{3} + \tfrac{4}{3} \right) - 0$$

$$= \frac{\pi}{6}.$$

⑥ a) TRUE

b) FALSE : this is backwards, and one also needs to know both functions are ≥ 0 to use the Comparison Theorem.

c) FALSE:

$$\frac{d}{dx}\left(\frac{1}{3}\sin^3 x + C\right) = \sin^2(x)\cdot\cos x.$$

d) TRUE: this can be done with a combination of long division, partial fractions, completing the square, the power rule, logarithms, and arctangents.

e) FALSE: Consider the case

$f(x) = x, \quad g(x) = -x.$