① We first calculate the left-hand side (LHS) and the right-hand side (RHS) of the DE $\frac{dy}{dx} = 4xy$ for each of the functions listed.

	LHS($\frac{dy}{dx}$)	RHS($4xy$)
a) $y = e^{-4x}$	$e^{-4x}\cdot(-4) = -4e^{-4x}$	$4xe^{-4x}$
b) $y = 4x$	4	$4x \cdot 4x = 16x^2$
c) $y = e^{2x^2}$	$e^{2x^2}\cdot(4x) = \underline{4xe^{2x^2}}$	$\underline{4x\cdot e^{2x^2}}$
d) $y = -4x$	-4	$4x\cdot(-4x) = -16x^2$
e) $y = e^{2x}$	$e^{2x}\cdot 2 = 2e^{2x}$	$4x\cdot e^{2x}$
f) $y = 2x^2$	$4x$	$4x\cdot 2x^2 = 8x^3$
g) $y = 4e^{2x^2}$	$4e^{2x^2}\cdot(4x) = \underline{16xe^{2x^2}}$	$4x\cdot 4e^{2x^2} = \underline{16xe^{2x^2}}$
h) $y = 2e^{4x}$	$2e^{4x}\cdot 4 = 8e^{4x}$	$4x\cdot 2e^{4x} = 8xe^{4x}$

For the functions in c) and g), the LHS and RHS are equal. This means that the functions in c) and g) satisfy the DE $\frac{dy}{dx} = 4xy$. None of the other functions listed satisfy the DE, since computing $\frac{dy}{dx}$ and $4xy$ for each of them does not give the same values.

② Answer: the functions in b), e), i), and j) are solutions to the DE $y'' + y' = 6y$, and none of the others are solutions.
The Work: each function needs to be checked. We can calculate y', y'', and then see if $y'' + y'$ equals $6y$ for each of the given functions. If it does, then the function is a solution to the DE.

a) $y = e^{-4x}$, $y' = -4e^{-4x}$, $y'' = 16e^{-4x}$, $y'' + y' = 12e^{-4x} \neq 6e^{-4x}$ No

b) $y = e^{-3x}$, $y' = -3e^{-3x}$, $y'' = 9e^{-3x}$, $y'' + y' = 9e^{-3x} - 3e^{-3x} = 6e^{-3x}$ ✓ Yes

c) $y = e^{-2x}$, $y' = -2e^{-2x}$, $y'' = 4e^{-2x}$, $y'' + y' = 4e^{-2x} - 2e^{-2x} = 2e^{-2x} \neq 6e^{-2x}$ No

2d) $y = -4e^{-2x^2}$, $y' = -4e^{-2x^2}\cdot(-4x) = 16xe^{-2x^2}$,

$y'' = 16e^{-2x^2} + 16xe^{-2x^2}\cdot(-4x) = 16e^{-2x^2} - 64x^2e^{-2x^2}$,

$y'' + y' = 16e^{-2x^2} - 64x^2e^{-2x^2} + 16xe^{-2x^2} \neq 6(-4e^{-2x^2}) = -24e^{-2x^2}$ No

	y	y'	y''	$y''+y'$	$6y$	Equal?
e)	e^{2x}	$2e^{2x}$	$4e^{2x}$	$4e^{2x}+2e^{2x}$	$6e^{2x}$	Yes ✓
f)	$2x^2$	$4x$	4	$4+4x$	$12x^2$	No
g)	$2\sin(2x)$	$4\cos(2x)$	$-8\sin(2x)$	$-8\sin(2x)+4\cos(2x)$	$12\sin(2x)$	No
h)	$4e^{-4x}$	$-16e^{-4x}$	$64e^{-4x}$	$64e^{-4x}-16e^{-4x}$	$24e^{-4x}$	No
i)	$3e^{-3x}$	$-9e^{-3x}$	$27e^{-3x}$	$27e^{-3x}-9e^{-3x}$	$18e^{-3x}$	Yes ✓
j)	$3e^{2x}$	$6e^{2x}$	$12e^{2x}$	$12e^{2x}+6e^{2x}$	$18e^{2x}$	Yes ✓

③ The matches are: i) b , ii) none , iii) d , iv) a , v) c

Explanation: The slope field in a) only has positive slopes, so it must match the DE in iv), $y' = y^2 + 1$. It can't match any of the other DE's since each of those has $y' = 0$ and $y' < 0$ for certain values of x and y.

The DE in ii), $y' = \sin x$, depends on x and does not depend on y. Therefore, its slope field will consist of identical slopes at a given x-value (regardless of the y-value). E.g., . None of the slope fields has this property, so equation ii) has no matching slope field. ☹

For the DE in v), $y' = \sin y$, the slopes should be zero for $y = 0$, $y = \pi$, and $y = -\pi$. This is only true in c). Also, in the other slope fields, the slopes change if we fix y and let x change. But $y' = \sin y \Rightarrow$ the slopes y' only depend on y (e.g., ). So the other slope fields (besides c)) cannot match the DE in v).

For the DE in i), $y' = xy + 1$, we see that the slope at the origin is $y' = 0 \cdot 0 + 1 = 1$. This rules out the slope fields in c) and d). Since $xy + 1 = 0 \Rightarrow y = -\frac{1}{x}$, the slopes should be zero along the curves for $y = -\frac{1}{x}$ (), e.g., at points like $(1,-1)$ and $(2,-\frac{1}{2})$.

This matches the behavior in b). It also rules out the slope field in a), since that one only has positive slopes.

The DE in iii), $y' = xe^{-y}$, has slopes equal to zero when $xe^{-y} = 0$, so, when $x = 0$ (since $e^{-y} \neq 0$). This rules out all of the slope fields except possibly d). Since the slope field is only approximate, its possible that the one represented in d) actually has slopes equal to zero when $x = 0$. (In fact, it does.) Another feature to check is which regions should have $y' > 0$. For $y' = xe^{-y}$, $y' > 0$ when $x > 0$, and $y' < 0$ when $x < 0$. This matches the slope field in d). Also, when y is a large positive value, then xe^{-y} is close to zero, which also matches the picture in d). We can check other features of the DE, but they all match the slope field in d), so d) is the answer.

④ a) The solution curves are increasing for $-1 < y < 2$. This is the only region for which solution curves are increasing, since the slope y' is zero when $y=-1$ or $y=2$, and $y' = 0.5(1+y)(2-y)$ is negative if $y<-1$ or if $y>2$.

b) The solution curves tend toward a finite y-value as $x \to \infty$ in the region where $y \geq -1$. When y is a large number ($y>2$), then the slope $y' = 0.5(1+y)(2-y)$ is negative, which means the solution curve must decrease. But when such a solution curve gets closer to $y=2$, its slope gets close to zero, which prevents it from decreasing without bound. Similarly, curves in the region $-1 < y < 2$ are increasing since $y' > 0$ there, but if such curves get too close to $y=2$, their slopes must get very close to zero, which prevents them from increasing without bound. Curves in the region $y < -1$ are decreasing since y' is negative when $y < -1$. In fact, the more negative y is, the more negative $y' = 0.5(1+y)(2-y)$ is. Thus curves in this region decrease without bound.

Finally, if a curve starts with $y=-1$ or $y=2$, then $y' = 0.5(1+y)(2-y)$ will be zero. This means there is no change in y with respect to x, so the y-coordinate stays the same even when the x-coordinate increases. If the y-coordinate stays the same, then the solution curve is horizontal ($y=-1$ or $y=2$), which means it tends toward a finite y-value (-1 or 2) as $x \to \infty$.

⑤ The answer is c). The differential equation (DE) in a) cannot be correct because $\frac{dy}{dx} = y - x$ implies that the slopes should be zero when $y - x = 0$, or when $y = x$. But the slope at (1,1) is definitely positive. For b), $\frac{dy}{dx} = y^2 - x^2 = (y-x)(y+x)$, so the slopes should again be zero when $y - x = 0$, or when $y = x$. For d), $\frac{dy}{dx} = y^2 + x^2$ implies that the slopes should never be negative, but they are, for example, at (-1,0) and (0,-1). For e), $\frac{dy}{dx} = y - x^2$, so the slopes should be zero when $y - x^2 = 0$, or $y = x^2$. But the slope is not zero at (1,1), which is on the curve $y = x^2$. For f), $\frac{dy}{dx} = x - y^2$, so the slopes should be zero when $x = y^2$, or when $y = \pm\sqrt{x}$. Again, this is not the case, since the slope at (1,1) is not zero.

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⑥ $\frac{dy}{dx} = \frac{x + \sin x}{3y^2} \Rightarrow 3y^2\,dy = (x + \sin x)\,dx \Rightarrow \int 3y^2\,dy = \int (x + \sin x)\,dx$

$\Rightarrow y^3 = \frac{1}{2}x^2 - \cos x + C \Rightarrow \underline{y = \sqrt[3]{\frac{1}{2}x^2 - \cos x + C}}$

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⑦ We start with $\frac{dy}{dx} = 4 - 7y$, and use the method of separable variables to solve for y.

$$\frac{1}{4-7y}\,dy = dx \Rightarrow \int \frac{1}{4-7y}\,dy = \int dx$$

Let $u = 4 - 7y$. Then $du = -7\,dy$, so $-\frac{1}{7}\,du = dy$, and

$$\int \frac{1}{4-7y}\,dy = -\frac{1}{7}\int \frac{1}{u}\,du = -\frac{1}{7}\ln|u| + C_1 = -\frac{1}{7}\ln|4-7y| + C_1$$

The right-hand side is $\int dx = x + C_2$, so we get

(⑦ continued)

$-\frac{1}{7}\ln|4-7y|+C_1 = x+C_2$, $-\frac{1}{7}\ln|4-7y| = x+C_3$,

$\ln|4-7y| = -7x+C_4$, $|4-7y| = e^{-7x+C_4}$,

$4-7y = \pm e^{C_4}e^{-7x} = C_5e^{-7x}$, $-7y = C_5e^{-7x}-4$,

$y = C_6e^{-7x}+\frac{4}{7}$. Since $y(0)=3$, we get

$3 = C_6e^0+\frac{4}{7} \Rightarrow 3-\frac{4}{7} = C_6 \Rightarrow C_6 = \frac{17}{7}$. So the

solution is $\underline{y = y(x) = \frac{17}{7}e^{-7x}+\frac{4}{7}}$.

⑧

ⓐ $z(x) = x^2\cdot y(x) \Rightarrow z'(x) = 2x\cdot y(x) + x^2\cdot y'(x)$ (using the product rule).
Since $y(x)$ satisfies the diff. eqn. $xy'+2y=\cos(x^2)$, we know that
$x\cos(x^2) = x\cdot(xy'+2y) = x^2\cdot y'+2xy$. But this is exactly what $z'(x)$ equals.
So then we know that $z'(x) = x^2y'+2xy = x\cos(x^2)$. In other words,
$z(x)$ satisfies the diff. eqn. $z' = x\cos(x^2)$.

ⓑ $\frac{dz}{dx} = x\cos(x^2) \Rightarrow dz = x\cos(x^2)dx \Rightarrow z = \int x\cos(x^2)dx$.
Let $u=x^2$, then $du = 2xdx$ and $\frac{1}{2}du = xdx$. So

$$z(x) = \int x\cos(x^2)dx = \int \cos u\cdot\frac{1}{2}du = \frac{1}{2}\int\cos u\,du$$

$$= \frac{1}{2}\sin u + C = \boxed{\frac{1}{2}\sin(x^2)+C}.$$

(8c) Since y satisfies $xy'+2y=\cos(x^2)$, it follows that $z=x^2\cdot y(x)$ satisfies $z'=x\cos(x^2)$, according to part (a).
Thus, $z=\frac{1}{2}\sin(x^2)+C$ by part (b), so $x^2\cdot y(x)=\frac{1}{2}\sin(x^2)+C$.
Therefore, $y(x)=\frac{1}{2}\cdot\frac{1}{x^2}\sin(x^2)+\frac{C}{x^2}$.

Since $y(\sqrt{\pi})=0$, we should have

$$0=y(\sqrt{\pi})=\frac{1}{2}\cdot\frac{1}{\pi}\cdot\sin\pi+\frac{C}{\pi}$$
$$=0+\frac{C}{\pi}.$$

For this to be true, we must have $C=0$. So then, $\boxed{y(x)=\frac{\sin(x^2)}{2x^2}}$.

(9a) $y'=-ay\ln(y/b)=0 \Rightarrow y=0$ or $\ln(y/b)=0 \Rightarrow y/b=1 \Rightarrow y=b$.
Since y' does not exist if $y=0$, the only equilibrium solution of the equation is $\underline{y=b}$.

b) When $y>b$, $\ln(\frac{y}{b})$ is positive, so $\frac{dy}{dt}<0$ and so y will decrease. When $0<y<b$, $\ln(\frac{y}{b})<0$, so $\frac{dy}{dt}>0$, and y will increase. When $y<0$, the expression for $\frac{dy}{dt}$ is undefined. So we say that $\lim_{t\to\infty} y(t)=b$, which is $\underline{iii}$).

c) $\frac{dy}{dt}=-ay\ln(y/b) \Rightarrow \frac{1}{y\ln(y/b)}dy=-a\,dt \Rightarrow \int\frac{1}{y\ln(y/b)}dy=\int -a\,dt$
To do the first integral, let $u=\ln(y/b)$. Then $du=\frac{1}{(y/b)}\cdot\frac{1}{b}dy=\frac{1}{y}dy$, and
$\int\frac{1}{y\ln(y/b)}dy=\int\frac{1}{u}du=\ln|u|+C=\ln|\ln(y/b)|+C$. So

$$\ln|\ln(y/b)|=-at+C_1 \Rightarrow |\ln(y/b)|=e^{-at+C_1} \Rightarrow \ln(y/b)=\pm e^{C_1}e^{-at}$$

$$\Rightarrow \ln(y/b)=C_2e^{-at} \Rightarrow y/b=e^{C_2e^{-at}} \Rightarrow \underline{y=be^{C_2e^{-at}}}$$

(9d) If $y = be^{C_2 e^{-at}}$ then $y' = be^{C_2e^{-at}} \cdot \frac{d}{dt}(C_2 e^{-at}) = be^{C_2e^{-at}}(C_2 e^{-at}(-a))$,

so $y' = -abC_2 e^{-at} e^{C_2 e^{-at}}$. Now we use $y = be^{C_2e^{-at}}$ and compute:

$$-ay\ln(y/b) = -abe^{C_2e^{-at}} \cdot \ln\left(\frac{be^{C_2e^{-at}}}{b}\right) = -abe^{C_2e^{-at}} \cdot \ln e^{C_2e^{-at}}$$

$$= -abe^{C_2e^{-at}} \cdot (C_2 e^{-at}).$$

Since this is the same as the y' we computed in part d), the answer from part c) must be correct.

(10a) If $V = \frac{1}{3}\pi\left(\frac{R}{H}h\right)^2 h = \frac{1}{3}\pi \cdot \frac{R^2}{H^2} \cdot h^3$, then $\frac{dV}{dt} = \pi \cdot \frac{R^2}{H^2} h^2 \cdot \frac{dh}{dt}$ by implicit differentiation. If $\frac{dV}{dt}$ is proportional to h with constant of proportionality k, (note that k is negative, since V is decreasing in time), we write

$$\frac{dV}{dt} = k \cdot h, \quad \text{so}$$

$$\pi \cdot \frac{R^2}{H^2} \cdot h^2 \frac{dh}{dt} = k\,h, \quad \text{which we rewrite as}$$

$$\boxed{\frac{dh}{dt} = \frac{k \cdot H^2}{\pi \cdot R^2} \cdot \frac{1}{h}}.$$

(b) Separating the variables, we get $h\,dh = \frac{kH^2}{\pi R^2}\,dt$, so

$$\frac{1}{2}h^2 = \frac{kH^2}{\pi R^2}t + C_1, \quad \text{so}$$

$$h = \sqrt{\frac{2kH^2}{\pi R^2}t + C} \quad \text{for some constant } C.$$

To determine C, we use the fact that $h(0) = H$ (i.e., the tank starts out full). Thus, $H = \sqrt{C}$, so $C = H^2$.

Our solution becomes

$$\boxed{h(t) = \sqrt{\frac{2kH^2}{\pi R^2}t + H^2}}.$$

(10c) Set $h(t)=0$, solve for t: $0=\sqrt{\frac{2kH^2}{\pi R^2}t+H^2}$,

so $-H^2=\frac{2kH^2}{\pi R^2}t$,

meaning $\boxed{t=-\frac{\pi R^2}{2k}}$.

(11a) Since carbon-14 undergoes exponential decay, the amount of carbon-14 present t years after harvest is given by $C(t)=C_0e^{-kt}$, where C_0 is the amount of carbon-14 present at the moment of harvest. We can find k using the half-life information: $\frac{1}{2}C_0=C_0e^{(-k)(5730)} \Rightarrow$

$\frac{1}{2}=e^{-5730k} \Rightarrow \ln(.5)=-5730k \Rightarrow k=\frac{\ln(.5)}{-5730}\approx 0.00012097$. So our model is $C(t)=C_0e^{-0.00012097t}$. For the lettuce leaf, we have

$.99999536\cdot C_0=C_0e^{-0.00012097t} \Rightarrow \ln(.99999536)=-0.00012097t$
$\Rightarrow t=\frac{\ln(.99999536)}{-0.00012097}\approx .038357$ years $\approx$ 14 days

(b) A one week old leaf has $C(\frac{7}{365})=C_0e^{(-0.00012097)(7/365)}\approx .99999768\,C_0$, or 99.999768% as much carbon-14 as a freshly cut leaf.

(12) $3P_0=P_0e^{k\cdot 20} \Rightarrow \ln 3=k\cdot 20 \Rightarrow k=\frac{\ln 3}{20}\approx .05493$

$2P_0=P_0e^{.05493\cdot t} \Rightarrow \ln 2=.05493t \Rightarrow t\approx 12.618$ years

(13) $\frac{dP}{dt}=kP \Rightarrow \int\frac{1}{P}dP=\int k\,dt \Rightarrow \ln|P|=kt+C_1 \Rightarrow |P|=e^{C_1}e^{kt} \Rightarrow$ $P=C_2e^{kt}$. Let $t=$ # of years since 1970. Then $P(t)=100e^{kt}$, so $900=100e^{k\cdot 10} \Rightarrow \ln 9=k\cdot 10 \Rightarrow k=\frac{\ln 9}{10}$. In 1995, we expect $P(25)=100e^{(\ln 9/10)\cdot 25}$ = 24,300 groundhogs.

(14) Since Carbon 14 undergoes exponential decay, we know that the amount, $y(t)$, present after t years is given by $y(t) = y_0 e^{-kt}$, where k is a constant and y_0 is the amount initially present. Since the half-life is 5730 years, we know that $\frac{1}{2} y_0 = y_0 e^{-k\cdot 5730}$, so $\frac{1}{2} = e^{-k\cdot 5730}$, $\ln \frac{1}{2} = -k\cdot 5730$, and $k = -\frac{\ln(\frac{1}{2})}{5730} \approx 0.00012097$. For the island of Hawaii, we know that $\frac{y(100{,}000)}{y_0} = e^{-\left(\frac{\ln 2}{5730}\right)\cdot 100{,}000} \approx \underline{0.0000055762}$.

(15) The logistic equation for a function $P(t)$ is $\frac{dP}{dt} = kP(1-\frac{P}{K})$, where k is the initial relative growth rate and K is the carrying capacity. So the initial value problem here is

$$\underline{\frac{dP}{dt} = .05\,P\left(1-\frac{P}{10{,}000}\right), \quad P(0) = 5700}$$

(16) We have $\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right)$, where $k = 0.25$.

The solution to the equation is $P(t) = \frac{K}{1+Ae^{-0.25t}}$, where A is a constant.

In fact, $A = \frac{K-P(0)}{P(0)}$ (this can be verified by setting $t=0$ above and solving for A), and since $P(0) = 0.01K$, we have

$$A = \frac{K-0.01K}{0.01K} = \frac{0.99K}{0.01K} = 99.$$

We want t such that $P(t) = \frac{1}{2}K$, so we solve the equation below for t:

$$\frac{1}{2}K = \frac{K}{1+99e^{-0.25t}} \Rightarrow \frac{1}{2} = \frac{1}{1+99e^{-0.25t}}$$

$$\Rightarrow 2 = 1+99e^{-0.25t}$$

$$\Rightarrow 1 = 99e^{-0.25t}$$

$$\Rightarrow \frac{1}{99} = e^{-0.25t} \Rightarrow t = \frac{\ln(1/99)}{-0.25} = \boxed{\frac{\ln(99)}{0.25}} \text{ (months)}.$$

(17) (a) The term -6 in the rate of change means that 6 lemons are harvested each month. (The growth rate of lemons is governed by the logistic equation but is diminished by this harvesting amount, 6 lemons/month.)

(b) Equilibrium solutions occur where $\frac{dP}{dt}=0$, so set $\frac{1}{2}P(1-\frac{P}{50})-6=0$.

$$\frac{1}{2}P-\frac{P^2}{100}-6=0 \Rightarrow -\frac{1}{100}(P^2-50P+600)=0$$
$$\Rightarrow -\frac{1}{100}(P-30)(P-20)=0,$$

so the equilibrium solutions are $\boxed{P=20}$ and $\boxed{P=30}$.

(c) We'll make use of the above factorization for the expression that equals $\frac{dP}{dt}$.

If $\underline{P<20}$, then $P-20<0$, and $P-30<-10<0$, so
$-\frac{1}{100}(P-30)(P-20)<0$ (negative × negative × negative).
Thus, $\frac{dP}{dt}<0$ here, and $P(t)$ is <u>decreasing</u>.

If $\underline{20<P<30}$, then $P-20>0$ and $P-30<0$.
Thus, $-\frac{1}{100}(P-30)(P-20)>0$ (negative × negative × positive),
so $\frac{dP}{dt}>0$ here, and $P(t)$ is <u>increasing</u>.

If $\underline{P>30}$, then $P-20>10>0$, and $P-30>0$.
Thus, $-\frac{1}{100}(P-30)(P-20)<0$ (negative × positive × positive),
so $\frac{dP}{dt}<0$ here, and $P(t)$ is <u>decreasing</u>.

(17) (d) The increase in P is most rapid when the change in P is largest, so, when $\frac{dP}{dt}$ has a maximum. To find where $\frac{dP}{dt}$ has a maximum, we take its derivative and set it equal to zero. Since $\frac{dP}{dt} = \frac{1}{2}P - \frac{P^2}{100} - 6$, its derivative is

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\left(\frac{dP}{dt}\right) = \frac{d}{dt}\left(\frac{1}{2}P - \frac{P^2}{100} - 6\right)$$

$$= \frac{1}{2}\frac{dP}{dt} - \frac{2P}{100}\frac{dP}{dt} \qquad \text{(chain rule/implicit diff.)}$$

$$= \left(\frac{1}{2} - \frac{P}{50}\right)\frac{dP}{dt}$$

$$= \left(\frac{1}{2} - \frac{P}{50}\right)\left[-\frac{1}{100}(P-20)(P-30)\right]$$

(using the work we did in part (a)). Setting this equal to 0, we see that one factor must equal zero, so either $P=25$, $P=20$, or $P=30$.

If $P=25$, $\frac{dP}{dt} = \frac{25}{2}\left(1-\frac{1}{2}\right) - 6 = \frac{1}{4}$;

if $P=20$, $\frac{dP}{dt} = \frac{20}{2}\left(1-\frac{2}{5}\right) - 6 = 0$;

if $P=30$, $\frac{dP}{dt} = \frac{30}{2}\left(1-\frac{3}{5}\right) - 6 = 0$. So $\frac{dP}{dt}$ has a maximum where $\boxed{P=25}$.

(e) We saw that if $P<20$, then $P(t)$ is decreasing. Thus, either $P \to -\infty$, or if we're restricting P to be positive (such as # of lemons), then $P \to 0$.

If $20<P<30$, then $\frac{dP}{dt}>0$, so P will increase, but as P gets closer to 30, $\frac{dP}{dt}$ gets closer to zero. (The curve $P(t)$ flattens as it rises.) Thus, P will approach 30 as $t \to \infty$.

If $P>30$, then P is decreasing. But again, as P gets closer to 30, $\frac{dP}{dt}$ gets closer to zero. Thus, P will approach 30 as $t \to \infty$.

<u>Summary</u>: if P is initially less than 20, it decreases (to $-\infty$ or 0).

If P is initially greater than 20, then $P \to 30$ as $t \to \infty$.

(17)(f) We begin with $\frac{dP}{dt} = \frac{1}{2}P\left(1-\frac{P}{50}\right) - 6 = -\frac{1}{100}\left(P^2 - 50P + 600\right) = -\frac{1}{100}(P-20)(P-30)$.

Separating variables, we get $\frac{dP}{(P-20)(P-30)} = -\frac{1}{100}\,dt$, so

$$\int \frac{dP}{(P-20)(P-30)} = \int -\frac{1}{100}\,dt = -\frac{1}{100}t + C.$$

To find the left-hand integral, we use partial fraction decomposition:

$$\frac{1}{(P-20)(P-30)} = \frac{A}{P-20} + \frac{B}{P-30} \quad\Rightarrow\quad 1 = A\cdot(P-30) + B\cdot(P-20)$$

$$\Rightarrow \quad A+B=0 \text{ and}$$
$$-30A-20B=1.$$

Thus, $A=-B$, so that $30B-20B=1$, i.e. $B=\frac{1}{10}$, and $A=\frac{-1}{10}$.

So we have $\int \frac{dP}{(P-20)(P-30)} = \int\left(\frac{-1/10}{P-20} + \frac{1/10}{P-30}\right)dP = -\frac{1}{10}\ln|P-20| + \frac{1}{10}\ln|P-30|$,

so $-\frac{1}{10}\ln|P-20| + \frac{1}{10}\ln|P-30| = \underline{\frac{1}{10}\ln\left|\frac{P-30}{P-20}\right| = -\frac{1}{100}t + C}.$

Solving for P in terms of t, we get

$$\ln\left|\frac{P-30}{P-20}\right| = -\frac{1}{10}t + 10C$$

$$\Rightarrow \left|\frac{P-30}{P-20}\right| = e^{-\frac{1}{10}t + 10C}$$

$$\Rightarrow \frac{P-30}{P-20} = \pm e^{-\frac{1}{10}t + 10C} = \pm e^{10C}e^{-\frac{1}{10}t} = Ae^{-\frac{1}{10}t}$$

(where A is any positive or negative constant: $A = \pm e^{10C}$).

(continued)

(17)(f) (continued) Applying the initial condition, we set $t=0$ and $P=25$:

$$\frac{25-30}{25-20} = Ae^{0} \Rightarrow A = \frac{-5}{5} = -1,$$

so $\frac{P-30}{P-20} = -e^{-t/10}$. Solving for P in terms of t, we find

$$P-30 = -(P-20)e^{-t/10}$$
$$= -Pe^{-t/10} + 20e^{-t/10}$$
$$\Rightarrow \quad P + Pe^{-t/10} = 30 + 20e^{-t/10}$$
$$\Rightarrow \quad P\left(1+e^{-t/10}\right) = 30 + 20e^{-t/10}$$
$$\Rightarrow \quad \boxed{P(t) = \frac{30+20e^{-t/10}}{1+e^{-t/10}}}$$

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(18)(a) Equilibrium solutions occur when $\frac{dN}{dt} = 0$. By setting each factor equal to zero, we find $N=0$, $N=K$, or $N=a$.
(Note: the case $N=0$ is problematic, since the right-hand side of the diff. eqn. is undefined at $N=0$. However, with the interpretation of N as population density, it is clear that a population with size 0 will continue to have size 0 for all t -- this is an equilibrium condition. By the way, the original exam solutions didn't place such a fine point on this!)

(18)(b) We can rule out III, IV for the fact that these two direction fields depict a positive slope in the region where $0<N<a$.
(In actuality, according to the equation, if $0<N<a$, then $1-\frac{N}{K}$ is positive (and less than 1) and $1-\frac{a}{N}$ is negative (since $\frac{a}{N}>1$), meaning $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)\left(1-\frac{a}{N}\right)$ is negative.)

(continued)

(18)(b) ctd. We can rule out I for the fact that this direction field depicts a slope very close to zero where N is very close to 0. (In actuality, according to the equation, if N is near 0, then $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)\left(1-\frac{a}{N}\right) = r\left(1-\frac{N}{K}\right)(N-a) \approx r\cdot 1\cdot(-a) = -ar$, a negative constant.) Thus, the best answer is II.

(c) We consider what happens to $N(t)$ as $t\to\infty$, considering various initial values of N. If $0<N<a$, then we have

$$\frac{dN}{dt} = \underbrace{rN}_{\text{pos}}\underbrace{\left(1-\overset{\text{less than 1}}{\frac{N}{K}}\right)}_{\text{pos}}\underbrace{\left(1-\overset{\text{greater than 1}}{\frac{a}{N}}\right)}_{\text{neg}} < 0,\quad \text{so } N \text{ decreases.}$$

Since N is a population density, N can't decrease below 0. So if $0<N<a$, then N will decrease to 0. This means that $N=0$ is a stable equilibrium. If $a<N<K$, then

$$\frac{dN}{dt} = \underbrace{rN}_{\text{pos}}\underbrace{\left(1-\overset{\text{less than 1}}{\frac{N}{K}}\right)}_{\text{pos}}\underbrace{\left(1-\overset{\text{less than 1}}{\frac{a}{N}}\right)}_{\text{pos}} > 0,\quad \text{so } N \text{ increases.}$$

Since K is an equilibrium solution and $\frac{dN}{dt}\to 0$ as $N\to K$, we have that if $a<N<K$, then N will increase to K. So $N=K$ is a stable equilibrium.

We can also check that if $K<N$, then

$$\frac{dN}{dt} = \underbrace{rN}_{\text{pos}}\underbrace{\left(1-\overset{\text{more than 1}}{\frac{N}{K}}\right)}_{\text{neg}}\underbrace{\left(1-\overset{\text{less than 1}}{\frac{a}{N}}\right)}_{\text{pos}} < 0,\quad \text{so } N \text{ decreases (to } K).$$

So $N=0$ and $N=K$ are stable equilibrium solutions. (And $N=a$ is unstable: $N(t)$ will move away from the value a if it starts near a.)

(19) a) Population x is the prey, and population y is the predator. This is true because in the absence of the predator (i.e., when $y=0$), the x population will grow. And in the absence of the prey (i.e., when $x=0$), the y population will decrease. Also, interactions between x and y lead to a decrease in x $(-bxy)$ and to an increase in y $(+dxy)$.

b) The constant b should be larger than the constant d. The term $-bxy$ gives the rate at which x decreases, due to encounters between x and y. Whatever that rate equals, we expect the rate of increase in y to be smaller. This is because the predator should have to eat a lot of prey (x) in order to produce one new (baby) predator y. So x should decrease more quickly than y increases, which means that b should be larger than d.

(20) Populations v and w cooperate with each other. This is true because interactions between v and w lead to an increase in v ("$+0.0005vw$") and an increase in w ("$+0.003vw$").

Next, population y preys on population z, which in turn preys on population x. Start by looking at x: in the absence of z, the population x will grow, and interactions with pop. z lead to decreasing x. ("$-0.005xz$"). But for z, interactions with pop. x lead to increasing z ("$+0.0002xz$"). Note furthermore, in the absence of its prey, that the growth rate of z is negative (i.e., z will die out). We see a similar plus/minus relationship between y & z: in this case, though, interactions between y and z lead to an increase in pop. y and a decrease in pop z; thus y preys on z. (Both populations y and z would decrease in the absence of other species, as evidenced by the negative terms "$-0.004y$" and "$-0.03z$" in $\frac{dy}{dt}$ and $\frac{dz}{dt}$ respectively.)

21 (a) Set both $\frac{dR}{dt}=0$ and $\frac{dW}{dt}=0$:

$$0 = kR - aRW = R(k-aW) = R(2.4-0.4W)$$
$$0 = -rW + bRW = W(-r+bW) = W(-10+0.2R)$$

Thus, from the first equation, either $R=0$ or $W=6$. If $R=0$, then the second equation becomes $0 = W\cdot(-10)$, so $W=0$. On the other hand, if $W=6$, then the second equation becomes $0 = 6\cdot(-10+0.2R)$, so $R = 10/0.2 = 50$. So the two equilibrium solutions are $\boxed{(R,W)=(0,0)}$ and $\boxed{(R,W)=(50,6)}$.

(b) (We're assuming the disaster struck when $(R,W)=(50,6)$, and not when $(R,W)=(0,0)$, for otherwise the problem seems somewhat trivial!) We begin with the rabbit population $R=25$ and the wolf population $W=3$. At these population values,

$$\frac{dR}{dt} = R(k-aW) = 25(2.4-(0.4)(3)) = (25)(1.2) = 30 \left(\frac{\text{rabbits}}{\text{unit time}}\right)$$

and $$\frac{dW}{dt} = W(-r+bR) = 3(-10+(0.2)(25)) = (3)(-5) = -15 \left(\frac{\text{wolves}}{\text{unit time}}\right)$$

That is, the rabbits will experience an <u>increase</u> at a rate of 30/unit time, and the wolves will experience a <u>decrease</u> at a rate of 15/unit time.

(c) If $W=0$, then $\frac{dR}{dt} = kR$. The solution $R(t)$ to this diff. eqn. is the exponential: $R(t) = Ae^{kt}$. If we say that $R(0)=60$, then $A=60$ and the rabbit population is given over time as

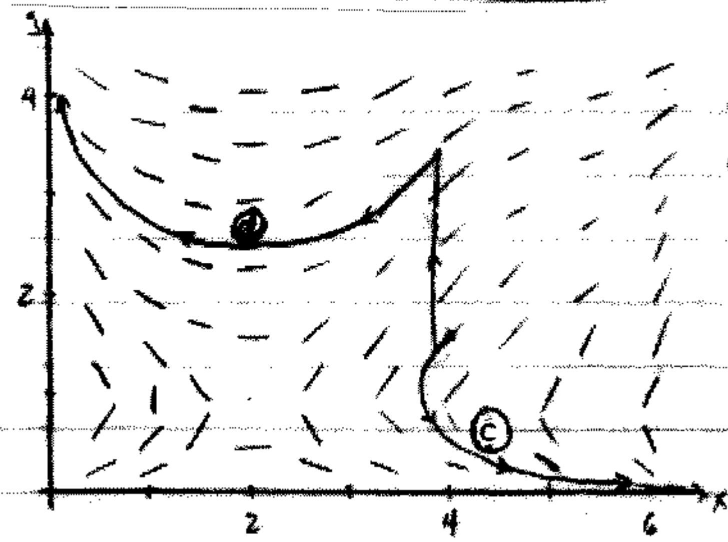
$$\boxed{R(t) = 60e^{2.4t}}.$$

(22) a) $x' = 0 \Rightarrow x - xy = x(1-y) = 0 \Rightarrow x = 0$ or $y = 1$

$y' = 0 \Rightarrow 2y - xy = y(2-x) = 0 \Rightarrow y = 0$ or $x = 2$

So the equilibrium solutions are $\underline{x=0 \text{ and } y=0}$ and $\underline{x=2 \text{ and } y=1}$, also denoted by $(0,0)$ and $(2,1)$.

b) $\frac{dy}{dx} = y'/x' = \underline{(2y-xy)/(x-xy)}$



c) To find the direction arrow at $(4, 3/2)$, we compute x' (or y') at $(4, 3/2)$. $x' = 4(1 - \frac{3}{2}) < 0$, so the curve is moving in the direction which decreases x.

d) In this case, before we have traveled very far from $(4, 3/2)$, we increase y by 2 without changing x.

Explanations: in part c), we can tell from the slope field that $y \to 0$ and x keeps increasing in the long run. This means that we expect one company (with net worth x) to dominate the market, and the other company to have a decreasing market share and perhaps go out of business. In part d), thanks to the timely influx of venture capitalist funds, we expect the fortunes of the two companies to be reversed in the long run. If we have enough data from the short term, we may be able to convince some VC's that the model presented here is correct, and hence they should give us millions of \$.