1.

If $y = \frac{\ln t + 3}{t}$, then $y(1) = \frac{\ln 1 + 3}{1} = 3$ ✓

and $y' = \frac{t\left(\frac{1}{t}\right) - (\ln t + 3)}{t^2} = \frac{-2 - \ln t}{t^2}$,

so $t^2 y' + t y = (-2 - \ln t) + (\ln t + 3)$
$= 1$ ✓

2. (5 points) Show that

$$y = \frac{1}{\sqrt{5 - t^2}}$$

is the solution of the initial value problem

$$y' = ty^3, \qquad y(1) = \frac{1}{2}.$$

The differential equation is separable, so we can solve it from scratch, but there's no need to do that here.

We just calculate:

$$y(1) = \frac{1}{\sqrt{5-1}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$$

and

2. ctd

$$y' = \frac{d}{dt}\left((5-t^2)^{-1/2}\right) = -\frac{1}{2}(5-t^2)^{-3/2}(-2t)$$

$$= (5-t^2)^{-3/2}\,t = \frac{t}{(5-t^2)^{3/2}}$$

$$= \left(\frac{1}{\sqrt{5-t^2}}\right)^3 \cdot t. \quad \checkmark$$

<u>Note</u>: we need to check that the differential equation $y' = ty^3$ is true for <u>all</u> t, not just when $t=1$.

3.

$$\frac{dy}{dt} = \frac{e^{-y^2}}{y} \cdot \frac{t+1}{t^2}$$

$$\int y e^{y^2}\,dy = \int \frac{t+1}{t^2}\,dt = \int (t^{-2}+t^{-1})\,dt$$

$$= -t^{-1} + \ln|t| + C$$

$u = y^2 \quad du = 2y\,dy$

$$\int y e^{y^2}\,dy = \frac{1}{2}\int e^u\,du = \frac{1}{2}e^u = \frac{1}{2}e^{y^2}.$$

$$\frac{1}{2}e^{y^2} = -\frac{1}{t} + \ln|t| + C.$$

3. ctd

$y(1) = 2 \Rightarrow$

$\frac{1}{2}e^4 = -1 + 0 + C \qquad C = 1 + \frac{1}{2}e^4.$

$e^{y^2} = -\frac{2}{t} + \ln|t| + 2C$

$y^2 = \ln\left[-\frac{2}{t} + \ln|t| + 2C\right]$

$y = \sqrt{\ln\left(-\frac{2}{t} + \ln|t| + 2 + e^4\right)}$

no $\pm$ since we know $y(1) = 2 > 0$.

Correction: The expression should read $\sqrt{\ln\left(-\frac{2}{t} + 2\ln|t| + 2 + e^4\right)}$; a coefficient "2" was omitted.

4. $\frac{dy}{dt} = te^{-t}y^2$

$\int \frac{1}{y^2}\,dy = \int te^{-t}\,dt$

$u = t \quad dv = e^{-t}dt$
$du = dt \quad v = -e^{-t}$

$\int te^{-t}\,dt$

$= -te^{-t} + \int e^{-t}dt = -te^{-t} - e^{-t} + C$

(watch the signs!)

4. ctd

$$= -e^{-t}(t+1) + C.$$

★ <u>Note</u>: C is a constant of integration that appears when you evaluate the integrals. If you just stick on $+C$ after solving for y you'll get the wrong answer!

So $$-\frac{1}{y} = -e^{-t}(t+1) + C.$$

$y(0)=2$: $$-\tfrac{1}{2} = -e^{0}(0+1) + C = -1 + C$$

$$\Rightarrow C = -\tfrac{1}{2} + 1 = \tfrac{1}{2}$$

$$-\frac{1}{y} = -e^{-t}(t+1) + \tfrac{1}{2}$$

$$\frac{1}{y} = e^{-t}(t+1) - \tfrac{1}{2} \Rightarrow y = \frac{1}{e^{-t}(t+1) - \frac{1}{2}}.$$

5 a) The natural growth gives us $\frac{1}{10}P$ millions of people added each year, but also $\frac{1}{10}\sqrt{P}$ million people are leaving each year.

So
$$P' = \frac{1}{10}P - \frac{1}{10}\sqrt{P} = \frac{1}{10}\sqrt{P}(\sqrt{P}-1).$$

b) Equilibrium is when P is constant so $P' = 0$.

$0 = \frac{1}{10}\sqrt{P}(\sqrt{P}-1)$ if $P = 0$ or $P = 1$.

So 0 or 1 million are equilibria.

c) Thinking about (but not taking the time to draw) the direction field,

slopes are positive if $P > 1$ and negative if $0 < P < 1$.

If $P_0 = \frac{1}{2}$, the population will go down to 0.

If $P_0 = 1$, the population will hold steady at 1 million.

If $P_0 = 4$, the population will increase forever.

d) $t_0 = 0 \quad P_0 = 4.$

$t_1 = 5 \quad P_1 = 4 + 5 \cdot \frac{1}{10}\sqrt{4}(\sqrt{4}-1)$
$= 4 + 1 = 5.$

$t_2 = 10 \quad P_2 = 5 + 5 \cdot \frac{1}{10}\sqrt{5}(\sqrt{5}-1)$
$= 5 + \frac{1}{2}(5 - \sqrt{5}) = \frac{5 - \sqrt{5}}{2}$ million people.

e) $\frac{dP}{dt} = \frac{1}{10}\sqrt{P}(\sqrt{P}-1)$

$$\int \frac{dP}{\sqrt{P}(\sqrt{P}-1)} = \int \frac{1}{10}\,dt = \frac{1}{10}t + C$$

$\Big\|$ $u = \sqrt{P}$ $\quad du = \frac{1}{2}P^{-1/2}dP = \frac{1}{2\sqrt{P}}\,dP$

$$\int \frac{2du}{u-1} = 2\ln|u-1| = 2\ln|\sqrt{P}-1|$$

So $\ln|\sqrt{P}-1| = \frac{1}{5}t + C$ ← (different C from before)

$$|\sqrt{P}-1| = e^{\frac{1}{5}t+C} = e^{C}e^{\frac{1}{5}t}$$

$$\sqrt{P}-1 = \pm e^{C}e^{\frac{1}{5}t} = Ae^{\frac{1}{5}t},\quad A = \pm e^{C}.$$

Now use $P(0) = 4$:

$$\sqrt{4}-1 = Ae^{0} \Rightarrow A = 1.$$

$$\sqrt{P}-1 = e^{\frac{1}{5}t}$$

$$\sqrt{P} = 1 + e^{\frac{1}{5}t}$$

$$P = \left(1 + e^{\frac{1}{5}t}\right)^2.$$

6. (20 points; 2 pages) A certain population of animals is dependent on a seasonally varying food supply. The rate at which the population grows is proportional to both the current population size P and to $\cos\left(\frac{\pi}{6}t\right)$; i.e., it is proportional to their product. (Here t is the time measured in months.)

Suppose the initial relative growth rate (i.e., $\frac{1}{P}P'$ when $t = 0$) is $\frac{1}{6}$ per month, and the initial population is 1000.

(a) Write a differential equation which models the growth of this population.

From the sentence "The rate at which..." we know

$$P' = kP\cos\left(\frac{\pi}{6}t\right), \quad k \text{ some constant.}$$

Since $\frac{1}{P}P' = \frac{1}{6}$ when $t=0$, we know

$$\frac{1}{6} = \frac{1}{P}P' = k\cos(0) = k, \quad \text{so}$$

$$P' = \frac{1}{6}P\cos\left(\frac{\pi}{6}t\right).$$

(b) Suppose the initial population is 1000. Use Euler's method with $h = 3$ to estimate the population after 12 months.

$t_0 = 0 \qquad P_0 = 1000$

$t_1 = 3 \qquad P_1 = P_0 + 3P'(t_0, P_0)$

$$= 1000 + 3\cdot\frac{1}{6}\cdot 1000\,\overbrace{\cos(0)}^{1}$$

$$= 1000 + 3\cdot\frac{1}{6}\cdot 1000 = 1500.$$

$t_2 = 6 \qquad P_2 = 1500 + 3\cdot\frac{1}{6}\cdot 1500\,\underbrace{\cos\left(\frac{3\pi}{6}\right)}_{0}$

$$= 1500$$

$t_3 = 9 \qquad P_3 = 1500 + 3\cdot\frac{1}{6}\cdot 1500\,\overbrace{\cos(\pi)}^{-1}$

$$= 1500 - 750 = 750$$

$t_4 = 12 \qquad P_4 = 750 + 3\cdot\frac{1}{6}\cdot 750\,\overbrace{\cos\left(\frac{9\pi}{6}\right)}^{0}$

$$= 750.$$

(c) Solve the differential equation to find an exact expression for the population after t months.

$$\frac{dP}{dt} = \frac{1}{6} P \cos\left(\frac{\pi}{6} t\right)$$

$$\int \frac{1}{P}\, dP = \int \frac{1}{6} \cos\left(\frac{\pi}{6} t\right) dt$$

$(P>0)$ $$\ln P = \frac{1}{\cancel{6}} \cdot \frac{\cancel{6}}{\pi} \sin\left(\frac{\pi}{6} t\right) + C$$

$$P = e^{\frac{1}{\pi}\sin\left(\frac{\pi}{6}t\right)} e^{C} = A e^{\frac{1}{\pi}\sin\left(\frac{\pi}{6}t\right)} \quad (A = e^C).$$

$$1000 = P(0) = A e^{\frac{1}{\pi}\sin(0)} = A \quad \text{so}$$

$$P = 1000\, e^{\frac{1}{\pi}\sin\left(\frac{\pi}{6}t\right)}.$$

1 (d) Find the exact population after 12 months, and compare to the Euler's method estimate.

$$P(12) = 1000\, e^{\frac{1}{\pi}\sin(2\pi)} = 1000 e^0$$

$$= 1000.$$

Euler's method gave an underestimate.

(7) a) $P' = 0$ if $P = 1000$, or $P = 200$.

Note: Technically speaking, the value $P = 0$ is not an equilibrium solution, because this value actually makes the expression for P' undefined (look at the last factor). But we'd accept this as a third equilibrium value, because in the real-world context of modeling population growth, a population of size zero will continue to stay of size zero as time progresses, and that's one definition of equilibrium.

b) $t_1 = 0 \qquad P_1 = 400$

$t_2 = 5 \qquad P_2 = P_1 + 5\,\frac{P_1}{10}\left(1 - \frac{P_1}{1000}\right)\left(1 - \frac{200}{P_1}\right)$

$$= 400 + 5 \cdot 40\left(1 - \frac{2}{5}\right)\left(1 - \frac{1}{2}\right)$$

$$= 400 + 200 \cdot \frac{3}{5} \cdot \frac{1}{2}$$

$$= 400 + 60 = 460.$$

$t_3 = 10 \qquad P_3 = 460 + 5 \cdot 460\left(1 - \frac{460}{1000}\right)\left(1 - \frac{200}{460}\right)$

$$= \underbrace{460 + 5 \cdot 460\left(\frac{540}{1000}\right)\left(\frac{260}{460}\right)}_{\approx P(10).}$$

8. (15 points) Suppose a population of fish in a lake grows according to a logistic model with carrying capacity 1000 and initial relative growth rate $k = 1/10$, but in addition 50 fish are caught per unit time.

(a) Write a differential equation that models the growth of this population of fish.

$$\frac{dP}{dt} = \frac{P}{10}\left(1 - \frac{P}{1000}\right) - 50.$$

Note: $\frac{dP}{dt}$ is a rate, and the rate at which fish are caught is $50 \frac{\text{fish}}{\text{unit of time}}$, so the last term is just -50, not $-50t$.

(b) Suppose the population at time $t = 0$ is 400. Use Euler's Method with $h = 2$ to estimate the population at time $t = 4$.

$t_1 = 0 \qquad P_1 = 400$

Make sure the 2 multiplies this whole thing!

$$t_2 = 2 \qquad P_2 = 400 + 2\left[\frac{400}{10}\left(1 - \frac{400}{1000}\right) - 50\right]$$

$$= 400 + 2\left[40 \cdot \frac{6}{10} - 50\right]$$

$$= 400 + 2[24 - 50] = 348.$$

$$t_3 = 4 \qquad P_3 = 348 + 2\left[\frac{348}{10}\left(1 - \frac{348}{1000}\right) - 50\right]$$

$$= 248 + 2 \cdot \frac{348}{10} \cdot \frac{652}{1000}.$$

9.

TRUE: since $y' < 0$.

TRUE: $P = K$ is the only nonzero equilibrium.

True / **False** All solutions of the differential equation $y' = 2 + \cos(ty)$ are increasing functions. Since $-1 < \cos\theta < 1$ always, here we know $y' > 2 - 1 = 1 > 0$, so y is increasing.

True / **False** If y is the solution of the initial value problem

$$y' = 3y\left(1 - \frac{y}{20}\right), \quad y(0) = 4$$

then $\lim_{t\to\infty} y(t) = 20$. Similar to Chapter 8 True-False Quiz #5.

This is a general fact we discussed about the logistic equation (see p. 539), or from the Formula Sheet:

$$\lim_{t\to\infty} \frac{20}{1+Ae^{-3t}} = \frac{20}{1+0} = 20 \qquad \left(A = \frac{20-4}{4} = 4\right).$$

True.

$$f'(t) = \frac{t\cdot\frac{1}{t} - \ln t}{t^2} = \frac{1-\ln t}{t^2},$$

so if $y = f(t)$,

$$t^2 y' + ty = (1-\ln t) + \ln t = 1 \checkmark.$$

10. (15 points) Solve the initial value problem

$$\frac{dy}{dt} = 3y - 2ty, \qquad y(0) = 5.$$

Show all of your work, with full mathematical justification.

$\frac{dy}{dt} = 3y - 2ty = y(3-2t)$, so by separation of variables,

$\frac{dy}{y} = (3-2t)\,dt$, and

$\int \frac{dy}{y} = \int (3-2t)\,dt$, so

$\ln|y| = 3t - t^2 + C.$

Thus $|y| = e^{3t-t^2+C} = e^C \cdot e^{3t-t^2}$, so

$y = e^C e^{3t-t^2}$ or $y = -e^C e^{3t-t^2}$.

But $y=5$ when $t=0$, so we must have $e^C = 5$, and thus

$\boxed{y = 5e^{3t-t^2}}$ is the only solution to the initial-value problem.

11. (12 points) The population of a species of elk on an Alaskan island has been observed to be closely predicted by a logistic model. The following specific observations have also been made:

- When the elk population was 600, the population was growing at a rate of 20 percent per year, or 120 elk per year.
- When the population was 800, the growth was 10 percent per year; i.e., 80 elk per year.

Find the carrying capacity of the population, giving complete mathematical justification.

The population function $P(t)$ in terms of time t (in years) satisfies

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{N}\right)$$

for some k and N.

But we know that if $P = 600$, then $\frac{dP}{dt} = 120$ $\left(\text{i.e. } \frac{1}{P}\frac{dP}{dt} = 0.20\right)$,

and if $P = 800$, then $\frac{dP}{dt} = 80$ $\left(\text{i.e. } \frac{1}{P}\frac{dP}{dt} = 0.10\right)$,

so we can say

$$120 = k \cdot 600 \cdot \left(1 - \frac{600}{N}\right) \quad \text{and}$$

$$80 = k \cdot 800 \cdot \left(1 - \frac{800}{N}\right).$$

Combining these, we find

$$\frac{0.20}{1 - \frac{600}{N}} = k = \frac{0.10}{1 - \frac{800}{N}},$$

so that $(0.20)\left(1 - \frac{800}{N}\right) = (0.10)\left(1 - \frac{600}{N}\right)$,

or $0.2 - \frac{160}{N} = 0.1 - \frac{60}{N}$.

Thus $0.1 = \frac{100}{N}$, so that $\boxed{N = 1000}$ (elks). (See next page for an alternate solution.)

Alternate solution. (more algebraically involved due to more unknown values)

The logistic equation $\frac{dP}{dt} = kP(1-\frac{P}{N})$ has general solution

$$P(t) = \frac{N}{1+Ae^{-kt}}, \quad \text{and thus } P'(t) = \frac{NAke^{-kt}}{(1+Ae^{-kt})^2} \quad \text{and} \quad \frac{P'(t)}{P(t)} = \frac{Ake^{-kt}}{1+Ae^{-kt}}.$$

Let's take $t=0$ for the time when $P=600$ and $P'=120$ and $\frac{P'}{P}=0.2$, and some $t=T$ for when $P=800$ and $P'=80$ and $\frac{P'}{P}=0.1$.

Thus,
$$600 = \frac{N}{1+A} \quad \text{and} \quad 0.2 = \frac{Ak}{1+A}, \quad \text{and}$$

$$800 = \frac{N}{1+Ae^{-kT}} \quad \text{and} \quad 0.1 = \frac{Ake^{-kT}}{1+Ae^{-kT}}.$$

Combining the first two equations, we find $\frac{N}{Ak} = \frac{600}{0.2} = 3000$, and using this fact together with the combination of the second two eqns, we find $\frac{N}{Ake^{-kT}} = \frac{800}{0.1}$, i.e. $e^{kT} = \frac{8000}{N/Ak} = \frac{8000}{3000} = \frac{8}{3}$.

It now follows that $600 = \frac{N}{1+A} \Rightarrow A = \frac{N}{600} - 1$,

as well as $800 = \frac{N}{1+Ae^{-kT}} \Rightarrow A = (e^{kT})(\frac{N}{800}-1) = (\frac{8}{3})(\frac{N}{800}-1)$,

so that $\frac{N}{600} - 1 = (\frac{8}{3})(\frac{N}{800}-1) = \frac{N}{300} - \frac{8}{3}$,

i.e. $\frac{N}{300} - \frac{N}{600} = \frac{8}{3} - 1$, so that

$$2N - N = 600 \cdot (\tfrac{8}{3}-1) = 600 \cdot \tfrac{5}{3} = 1000, \text{ i.e. } \boxed{N = 1000 \text{ elk}}.$$

12. (15 points) Two species, X and Y, compete with each other for the same limited resources. Their population sizes, $x(t)$ and $y(t)$, measured in the thousands and modeled as functions of time t (in years), obey the following system of differential equations:

$$\frac{dx}{dt} = 2x - xy,$$
$$\frac{dy}{dt} = 3y - xy.$$

(a) Find the equilibrium solutions of this system. Show your work.

Equilibrium solution is a pair $(x(t), y(t))$ where both members are constant fns, so $\frac{dx}{dt} = \frac{dy}{dt} = 0$. Solving

$$\begin{cases} 0 = 2x - xy = x(2-y) \\ 0 = 3y - xy = y(3-x) \end{cases},$$

we find that $x=0$ requires $y=0$, and $y=2$ requires $x=3$. (thousands)

Thus, the solutions are $\boxed{(x,y) = (0,0) \text{ and } (x,y) = (3,2).}$

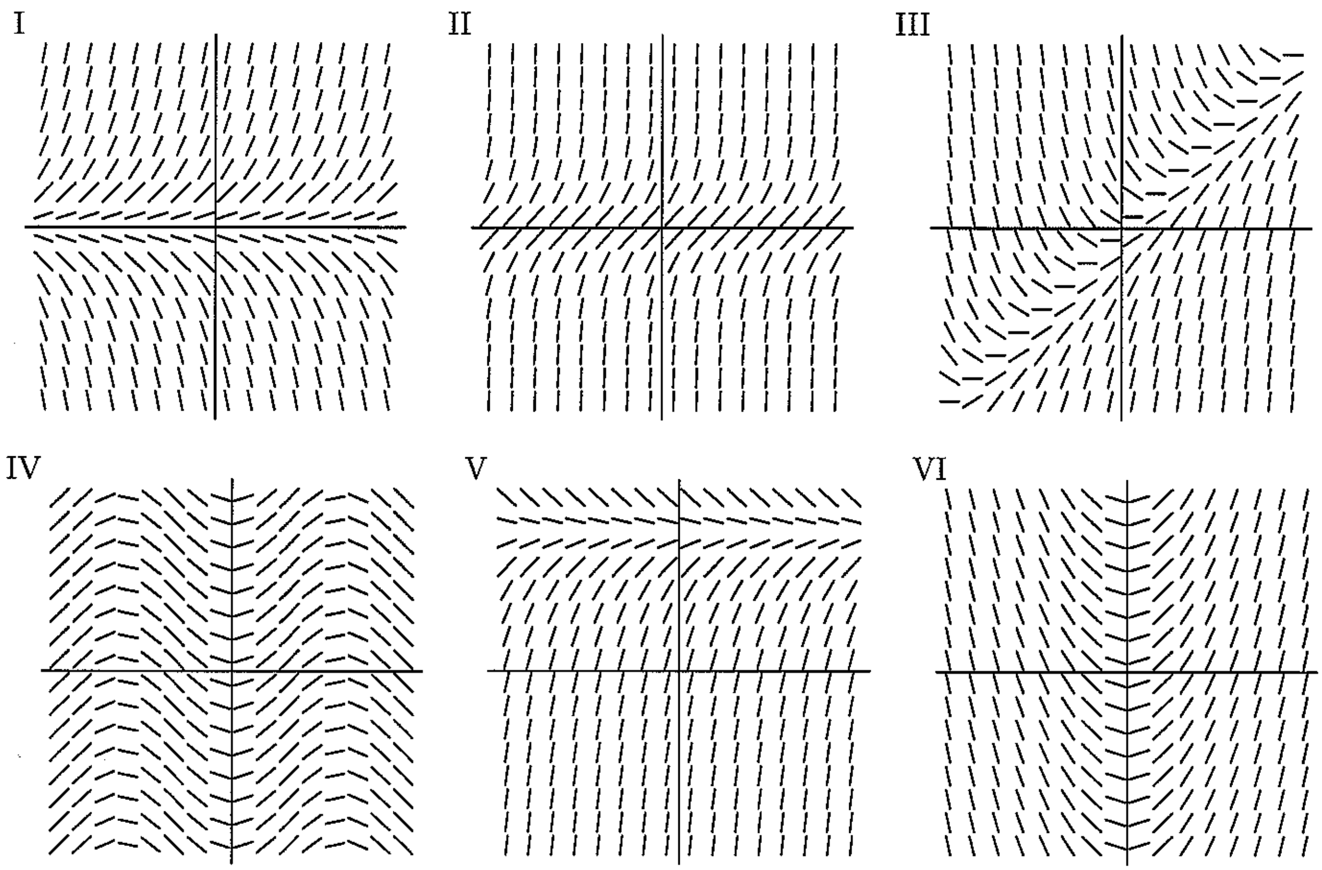
(b) At time $t = 0$ suppose $x(0) = 2$ and $y(0) = 4$. Find the rates of change of the two populations at this moment. What will happen in the long run, as t increases? Justify your answer.

At this moment, $\frac{dx}{dt} = 2\cdot 2 - 2\cdot 4 = -4 \ \frac{\text{thous}}{\text{yr}}$, and $\frac{dy}{dt} = 3\cdot 4 - 2\cdot 4 = 4 \ \frac{\text{thous}}{\text{yr}}$.

(In particular, species Y is growing and species X is diminishing.)

- Since $\frac{dx}{dt} = x(2-y)$, we note that $\frac{dx}{dt} < 0$ whenever $y > 2$ (and there remain any surviving species X).
- Meanwhile, $\frac{dy}{dt} = y(3-x)$, so $\frac{dy}{dt} > 0$ as long as $x < 3$ (and there exist a positive amount of species Y).
- Thus, after $t=0$, as y increases to well above 2 and x decreases to well below 3, the trend of "Y increasing, X decreasing" will continue to play out. It makes sense to expect X to die out eventually (or at least declare that "X approaches 0 while Y approaches ∞" mathematically).

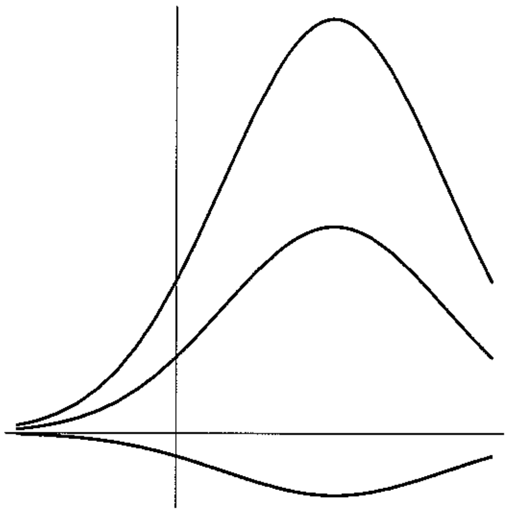
13. (18 points) Match the direction fields below with their differential equations, and give your reasoning. Each field is graphed for $-5 \le x \le 5$, $-5 \le y \le 5$.



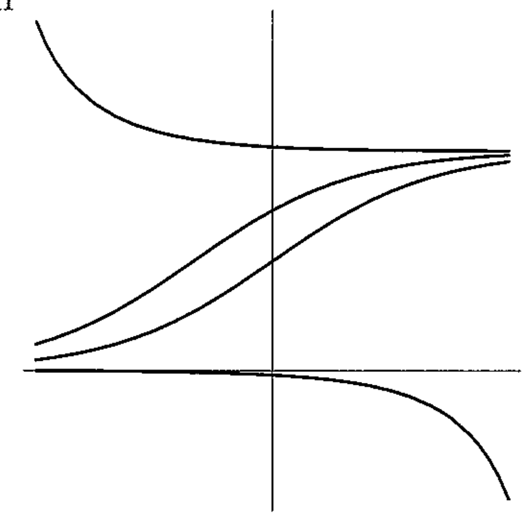
Equation	I, II, III, IV, V, or VI	Brief reason – *Sample answers only: others possible*
$dy/dx = 1 + y^2$	II	Equation & graph both have no dependence on x-variable (or horizontal position), and graphs I & V (which also have this property) can be eliminated because we also need dy/dx positive at all points.
$dy/dx = x$	VI	One of two equations/graphs to have no dependence on y-variable (or vertical position); can match IV to equation below and argue by process of elimination.
$dy/dx = \sin x$	IV	Only equation/graph to have periodicity in horizontal direction.
$dy/dx = y$	I	Slope negative for all points where $y<0$ and positive where $y>0$.
$dy/dx = x - y$	III	All other equations/graphs have dependence on only a single variable. In addition, graph III is unchanged along diagonal lines of the form $x-y=$ constant.
$dy/dx = 4 - y$	V	Slope negative if $y>4$ and positive if $y<4$.

14. (16 points) Each picture below depicts a few possible solution curves to a differential equation chosen from the list at the bottom of the page. Match each equation to its sketch of solutions. Here a, b, and k are fixed *positive* constants. The scales on the axes (t is horizontal; y is vertical) have been omitted because they don't affect the answers.

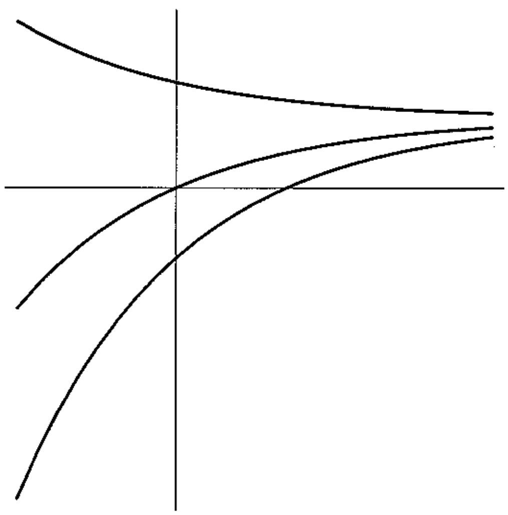
I



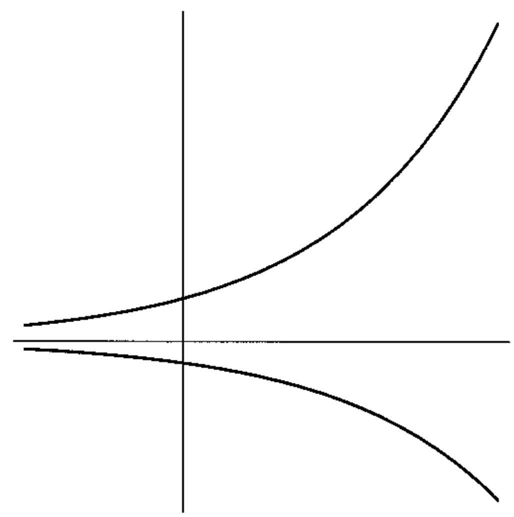
II



III



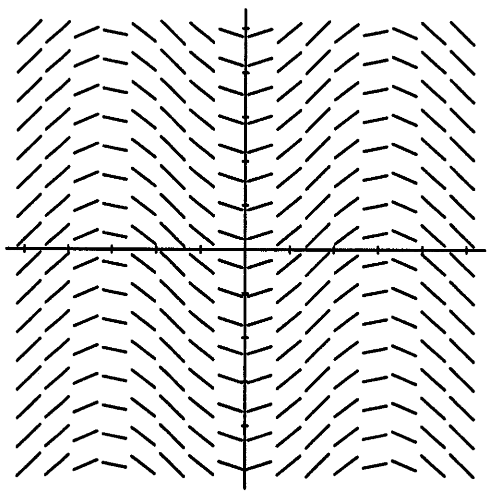
IV



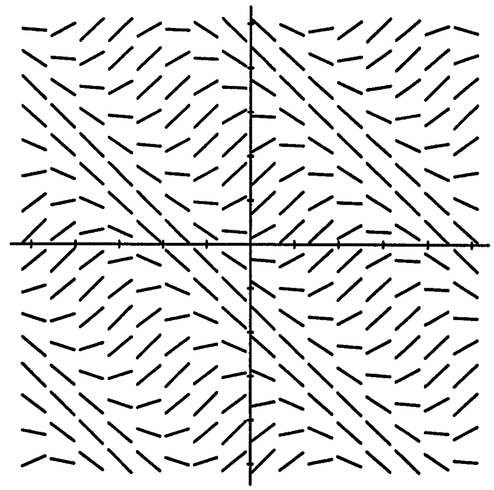
Equation	I, II, III, or IV	Brief reason – Samples
$dy/dt = ky$	IV	Equation has "exponential growth" function as solution; only IV has a curve fitting this description.
$dy/dt = (a - by)y$	II	This is the logistic equation, so we expect solutions reflecting a rise to some carrying capacity; II is only fit.
$dy/dt = a - by$	III	If $y > a/b$, we require slope to be negative, and if $y < a/b$, then slope must be positive. Only graph III can fit this requirement.
$dy/dt = (a - bt)y$	I	There is a single value of t (namely $t = \frac{a}{b}$) where any solution curve must be flat; graph I appears to fit.

15. (20 points) Match the slope fields below with their differential equations, and give your reasoning. Here k is a positive constant, and the scales on all axes are the same. (The horizontal variable is t; the vertical is y.)

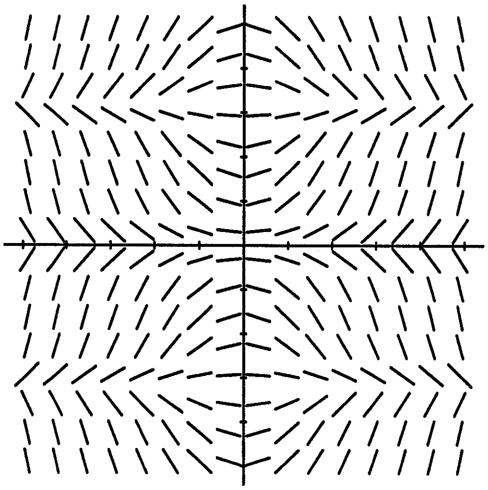
I



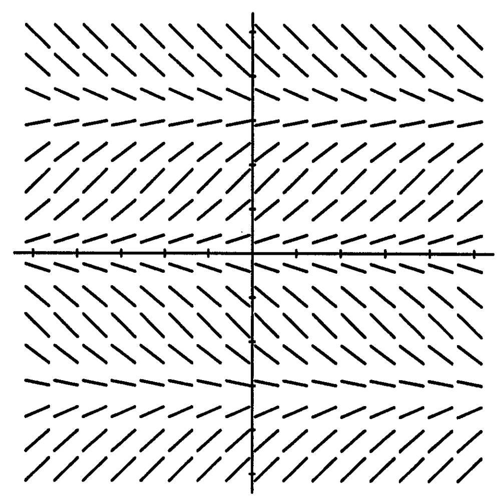
II



III



IV



Equation	I, II, III, IV, or "none"	Brief reason
$dy/dt = k\sin y$	IV	Graph IV has slopes equal when looking across a horiz. line -- its equation must be independent of t.
$dy/dt = k\sin t$	I	Graph I has slopes equal when looking along each vertical line -- its equation must be independent of y.
$dy/dt = kt\sin y$	III	Graph III has at least four lines along which all slope lines are 0: the horiz lines $y=\pi$, $y=0$, $y=-\pi$, as well as the vertical line $t=0$. This equation is the only one that is zero ~~for~~ all these areas.
$dy/dt = k\sin(ty)$	none	Could do this last & argue by elimination; alternatively notice that the equation is symmetric when t & y are interchanged, which rules out all graphs except II, and slope should be 0 when $t=0$, which rules out II.
$dy/dt = k\sin(t+y)$	II	Graph II has property that when $t+y$ is fixed constant, slope is the same: its equation must depend on $t+y$. Alternative: only graph with positive slope at both $(0,\pi/2)$ & $(\pi/2,0)$.

16. (20 points) The *Millenium Falcon*, widely believed to be the fastest spaceship in the galaxy, is trying to fly past the Death Star without getting pulled in by the Death Star's powerful tractor beam. The beam is activated at time $t = 0$, and the distance $x(t)$, in kilometers, between the spaceship and the Death Star after t seconds satisfies the differential equation

$$\frac{dx}{dt} = -7x(x - 100)(x - 10000).$$

(a) What is the physical meaning of "equilibrium solution" in this context? Find all equilibrium solutions.

Since "equilibrium solution" refers to solutions where $x(t)$ is constant, this corresponds to situations where the Falcon is maintaining a constant distance from the Death Star (so it isn't moving; i.e. it's standing still).

To find equilibrium solutions, set $x(t) = A$, so that $\frac{dx}{dt} = 0$, and plug in:

$$0 = -7A(A-100)(A-10000),$$

which means that the possible values for A are $\boxed{0,\ 100,\ \text{and } 10000}$. (km)

(These are the distances, in km, where the Falcon can maintain a constant distance from the Death Star.)

(b) For what positive values of x is x increasing? decreasing? Give complete answers.

Since $x(t)$ is increasing when $\frac{dx}{dt}$ is positive, and decreasing when $\frac{dx}{dt}$ is negative, we must check for which values of x is $\frac{dx}{dt} = -7x(x-100)(x-10000)$ positive or negative:

region	sign of $-7x(x-100)(x-10000)$
$0 < x < 100$	$\ominus \cdot \ominus \cdot \ominus = \ominus$
$100 < x < 10000$	$\ominus \cdot \oplus \cdot \ominus = \oplus$
$x > 10000$	$\ominus \cdot \oplus \cdot \oplus = \ominus$

Thus x is <u>increasing</u> when $100 < x < 10000$, and <u>decreasing</u> when $0 < x < 100$ or $x > 10000$.

(c) Predict the fate of the *Falcon* as t approaches infinity; your answer will likely depend on the initial distance $x_0 = x(0)$. That is, are there values of x_0 for which the ship escapes? Is sucked into the Death Star? Or some other possibility? A complete description should provide predictions that together cover every possible (positive) value of the initial distance x_0.

Here are the outcomes:

x_0 (kilometers)	Result as time approaches ∞
$x_0 = 0$ or 100 or 10000	Falcon holds steady at this distance for all t.
$0 < x_0 < 100$	Falcon is pulled into the Death Star.
$100 < x_0 < 10000$	Falcon pulls away to approach a distance of 10000 km. (Never gets further)
$x_0 > 10000$	Falcon is pulled in to a distance of 10000 km (but never pulled closer).

Justification (using parts ⓐ & ⓑ):

- The first case is just the set of equilibrium solutions from part ⓐ.
- If $0 < x_0 < 100$, then $\frac{dx}{dt} < 0$ at $t=0$, so the Falcon starts being pulled in. As its distance x decreases, then since x remains between 0 and 100, the derivative $\frac{dx}{dt}$ will continue to be negative, and x will continue to decrease; thus, the Falcon is pulled into the Death Star as t increases.
- If $100 < x_0 < 10000$, then $\frac{dx}{dt} > 0$ at $t=0$, so the Falcon will begin by pulling away from the Death Star. As its distance x increases and nears $x = 10000$, then although the sign of $\frac{dx}{dt}$ will continue to be positive, the size of $\frac{dx}{dt} = -7x(x-100)(x-10000)$ will become small, since $x - 10000$ will be small (near 0). Thus the Falcon will pull away at a slower and slower rate, approaching (asymptotically) the equilibrium distance of 10000 km as t increases.
- The reasoning for $x_0 > 10000$ is analogous to the previous case, but $\frac{dx}{dt}$ will be negative. Again, $\frac{dx}{dt}$ will be near 0 as x nears 10000, so the Falcon slows as it is pulled to a distance of $x = 10000$ km.

17. (20 points)

(a) The graph of a function $y = y(x)$ in the xy-plane has the property that the slope of the curve at each point is proportional to the y-coordinate of that point. Moreover, the curve passes through the point $(4, -9)$, and the value of the second derivative of the function at this point is -1. Find the function (or family of functions) with these properties.

$y(x)$ satisfies:

$$y'(x) = k \cdot y$$
$$y(4) = -9$$
$$y''(4) = -1$$

The general solution to the differential equation is $y(x) = Ce^{kx}$,

which means that $y'(x) = Cke^{kx}$ and
$y''(x) = Ck^2e^{kx}$.

Thus the initial conditions give

$$-9 = y(4) = Ce^{k \cdot 4} \quad \text{and}$$
$$-1 = y''(4) = Ck^2e^{k \cdot 4}.$$

Therefore $\frac{-9}{-1} = \frac{Ce^{k \cdot 4}}{Ck^2e^{k \cdot 4}} = \frac{1}{k^2}$, so $k = \pm\frac{1}{3}$.

If $k = \frac{1}{3}$, then $C = \frac{-9}{e^{4k}} = \frac{-9}{e^{4/3}}$, so $\boxed{y(x) = -\frac{9}{e^{4/3}} \cdot e^{x/3}}$,

and if $k = -\frac{1}{3}$, then $C = \frac{-9}{e^{-4/3}} = -9e^{4/3}$, so $\boxed{y(x) = -9e^{4/3} \cdot e^{-x/3}}$.

These are the only two solution curves.

(b) Find the general solution $y(t)$ to

$$\frac{dy}{dt} = \frac{1}{10}(10 - 3y)(10 + 3y).$$

(Hint: an integral table may be helpful, but it can be avoided by instead showing that the function $P(t) = 10 + 3y(t)$ satisfies a certain Logistic differential equation involving dP/dt.)

If $P(t) = 10 + 3y(t)$, then solving for y gives $y(t) = \frac{1}{3}P(t) - \frac{10}{3}$.

Thus $\frac{dy}{dt} = \frac{1}{3} \cdot \frac{dP}{dt}$, and

$$10 - 3y = 10 - 3\left(\tfrac{1}{3}P - \tfrac{10}{3}\right) = 20 - P, \text{ while}$$

$$10 + 3y = P.$$

The differential equation $\frac{dy}{dt} = \frac{1}{10}(10-3y)(10+3y)$ thus becomes

$$\frac{1}{3}\frac{dP}{dt} = \frac{1}{10}(20-P)\cdot P, \text{ or}$$

$$\frac{dP}{dt} = 6P\left(1 - \frac{P}{20}\right).$$

This is a logistic equation with $M=20$ and $k=6$, so the solution is

$$P(t) = \frac{20}{1+Ae^{-6t}} \quad \text{or} \quad P(t) = 0;$$ in terms of y this means

$$y(t) = \frac{1}{3}P(t) - \frac{10}{3} = \boxed{\frac{1}{3}\cdot\frac{20}{1+Ae^{-6t}} - \frac{10}{3} \quad \text{or } y = -10/3}.$$ (A is any constant)

Alternate solution to part (b), using separation of variables & integration:

$$\int \frac{dy}{(10-3y)(10+3y)} = \int \frac{1}{10}\,dt = \frac{t}{10} + C,$$ and left-hand side requires partial fractions.

If $\dfrac{1}{(10-3y)(10+3y)} = \dfrac{A}{10-3y} + \dfrac{B}{10+3y}$, then $1 = A(10+3y) + B(10-3y) = (10A+10B) + (3A-3B)y$,

so $\begin{matrix} 3A-3B=0, \\ 10A+10B=1 \end{matrix} \Rightarrow \begin{matrix} A=B \\ A+B=\frac{1}{10} \end{matrix} \Rightarrow A=B=\frac{1}{20}.$

Thus $$\int \frac{dy}{(10-3y)(10+3y)} = \int \left(\frac{1/20}{10-3y} + \frac{1/20}{10+3y}\right) dy = -\frac{1}{60}\ln|10-3y| + \frac{1}{60}\ln|10+3y| = \frac{1}{60}\ln\left|\frac{10+3y}{10-3y}\right|.$$

We get $\frac{1}{60}\ln\left|\frac{10+3y}{10-3y}\right| = \frac{t}{10} + C$, so solving for y yields

$$\left|\frac{10+3y}{10-3y}\right| = e^{6t+6C} = Be^{6t}$$ for $B>0$; thus,

$$\frac{10+3y}{10-3y} = Be^{6t}$$ for *any* $B \neq 0$ (pos or neg); finally we get

$$10+3y = (10-3y)\cdot Be^{6t} = 10Be^{6t} - (3Be^{6t})y,$$ so

$$y(3+3Be^{6t}) = 10Be^{6t} - 10,$$ and

$$y = \frac{10(Be^{6t}-1)}{3(Be^{6t}+1)}. \quad (B \neq 0).$$

Since separation of variables can miss equilibrium solutions, we separately note that

$\frac{dy}{dt} = 0$ implies $y = \frac{10}{3}$ or $y = -\frac{10}{3}$, so these are two other solutions.

Or, we note that the case $B=0$ covers the solution $y = -\frac{10}{3}$ already, so a complete solution is described by:

$$\boxed{y(t) = \frac{10}{3}\cdot\frac{Be^{6t}-1}{Be^{6t}+1}, \quad B \text{ any number, or } y(t) = \frac{10}{3}.}$$

(This is algebraically equivalent to the first solution; no simplification is needed beyond this.)

18. (15 points) On a strange tropical island, two native species, the mattababy and the henway, participate in a very predictable relationship. The mattababy population is described over time by a function $x(t)$, where t is measured in days; the henway population is described by a function $y(t)$. Naturalists coming to the island observe that the rates of growth of the two species are given by the equations

$$\frac{dx}{dt} = -4x + 0.02xy$$
$$\frac{dy}{dt} = -3y + 0.01xy$$

(a) Is the relationship between the two species one of mutual competition, mutual cooperation, or predator-prey (and if the latter case, which plays which role)? Justify your answer by explaining how the equations show what effect each species has on the other.

In the presense of $y=0$ henways, the mattababy population obeys $\frac{dx}{dt} = -4x$, which is exponential decay. But with a positive y, the growth rate of mattababies ($\frac{dx}{dt}$) increases, due to the "$+0.02xy$" term. Thus henways have a positive effect on the mattababy growth rate. Similarly, the "$+0.01xy$" term in the henway growth rate $\frac{dy}{dt}$ indicates that mattababies have a positive effect on henways. It follows that this is a relationship of <u>mutual cooperation</u>.

(b) Find all equilibrium solutions to the above system. Show your work.

Equilibrium solutions when $\frac{dx}{dt} = \frac{dy}{dt} = 0$, so

$$0 = -4x + 0.02xy = -4x\left(1 - \frac{0.02}{4}y\right) \quad \text{and}$$
$$0 = -3y + 0.01xy = -3y\left(1 - \frac{0.01}{3}x\right).$$

Thus $x=0$ or $1 - \frac{0.02}{4}y = 0$, i.e. $x=0$ or $y=200$. If $x=0$, then we must have $0 = -3y\left(1 - \frac{0.01}{3}\cdot 0\right) = -3y$, so $y=0$.

If instead $y=200$, then $0 = -3\cdot 200\cdot\left(1 - \frac{0.01}{3}x\right)$, so $x = 300$.

Thus the solutions are $\boxed{(x,y) = (0,0)}$ and $\boxed{(x,y) = (300,200)}$.