Math 41 — Strategies for Finding Absolute Extrema

There are two stages to solving a real-world optimization problem: first, convert the "word" question into a math problem involving absolute extrema; second, use the power of single-variable calculus to resolve this math portion. The steps we discussed in class (and the tips found in Section 4.6, pages 299-300) are designed to help you think through the first stage. The purpose of this sheet is to review everything we've seen in this course that can help you with the second stage, and to point you to places where the text uses these ideas.

The Setup

So the problem at hand is that we're faced with a *continuous* function $Q(x)$, with some domain D (let's say it's an *interval*, though not necessarily the entire x-axis), and we need to find the absolute maximum or minimum value of $Q(x)$ on D — or maybe the x-coordinate that produces this absolute max/min, depending on how the question is phrased.

If the domain D is a closed interval $[a, b]$: Closed Interval Method

In this case, we know that since $Q(x)$ is continuous, $Q(x)$ does have both an absolute max and absolute min on D (this is the Extreme Value Theorem); candidates for locations of absolute extrema include all the critical numbers of Q and both the endpoints ($x = a$ and $x = b$) of D. As in Section 4.2, build a list of candidates, and test the value of Q at each, ultimately picking the largest and smallest Q-values on this list. See this used in **Section 4.6, Examples 1, 4, and 5**.

What if the domain isn't a closed interval? Consider the situation!

(Examples: Say $D = (-\infty, \infty)$, or $D = (0, \infty)$, or $D = (-1, 1)$.)

You first need to be very skeptical: is there some reason to expect that there really *is* an absolute min, or max? There doesn't have to be. For example, if $Q(x) \to -\infty$ as x ranges through values in the domain, then Q has no absolute minimum. Many other things can go wrong, too, especially if you start considering functions Q with discontinuities. (Think of all the various examples drawn in Section 4.2.)

But you can guess that if an optimization problem demands a minimum value, then there probably is one. Furthermore, we are very often in simplified situations, where not a lot is happening (e.g. few critical numbers). Be ready to recognize what you can do in these situations. Here are a few cases that we see a lot (and that are not mutually exclusive):

Special Case #1: First Derivative Test for Absolute Extrema (page 302). If a function $Q(x)$ that is continuous on an interval D has critical number c, and if $Q'(x)$ is such that both $Q'(x) < 0$ for *all* $x < c$ in D, and $Q'(x) > 0$ for *all* $x > c$ in D, then Q has its absolute minimum at $x = c$. (Switch the signs of Q' to get a statement about the maximum.) See this used in **Section 4.6, Examples 2 and 3**, and in the solution to **Problem 11** (to be handed out separately).

Special Case #2: Second Derivative Test for Absolute Extrema. If a function $Q(x)$ that is continuous on an interval D has critical number c, and if $Q''(x) > 0$ for all x in D, then Q has its absolute minimum at $x = c$. (Switch the sign of Q'' to get a statement about the maximum.) See this used, but not by name, in **Section 4.6, Example 1** (just before the end, on page 301), and in the solution to **Problem 23** (to be handed out separately).

Special Case #3: Growth Test. If a function $Q(x)$ that is continuous on an interval D has *only one* critical number c, and if $Q(x) \to +\infty$ as x moves through D to the right of c, and if $Q(x) \to +\infty$ as x moves through D to the left of c, then Q has its absolute minimum at $x = c$. (Switch both signs of infinity to get a statement about the maximum.) See this used, but not by name, in **Section 4.6, Example 2 (Figure 5, page 301)**.

Special Case #4: A "Local to Global" Principle. If a function $Q(x)$ that is continuous on an interval D has *only one* critical number c, and if you can successfully use some method (e.g., from Section 4.3) to show that Q has a *local* minimum at $x = c$, then Q automatically has its absolute minimum at $x = c$. (Replace minimum with maximum for the analogous result.) This is implicitly being used in several of the special cases and text examples cited above.

Keep in mind — there may be more than one valid way to reason, just as there is in the case of characterizing local extrema (think about the choice you sometimes had in Section 4.3 between using the First or Second Derivative Test). In deciding how to proceed, try to think about what you can easily deduce from what you know about Q. (For instance, if it's easy to find $Q''(x)$, think about how you might use its sign to tell you something; alternatively, if it's easy to detect the sign of $Q'(x)$, use it to your advantage!)

As always, practice every sample problem, read every solution writeup, and look for patterns in the process; the experience gained will help you develop an understanding for how you can best attack a new situation.