Solutions and commentary to problems #11, 19, 23 from Section 4.6

Practical Tips for Modeling Optimization Problems

1. Make sure that you know what quantity or function is to be optimized.
2. If possible, make several sketches showing how the elements that vary are related. Label your sketches clearly by assigning variables to quantities which change.
3. Try to obtain a formula for the function to be optimized in terms of the variables that you identified in the previous step. If necessary, eliminate from this formula all but one variable. Identify the domain over which this variable varies.
4. Find the critical points and evaluate the function at these points and the endpoints (if relevant) to find the global (absolute) maxima and minima. (You must justify why your critical point or endpoint is the max/min you seek.)

Solution to problem 11:

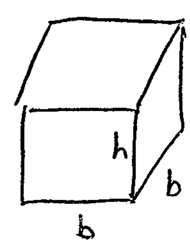
11. Let b be the length of the base of the box and h the height. The surface area is $1200 = b^2 + 4hb \Rightarrow h = (1200 - b^2)/(4b)$.

Since we require $b, h \geq 0$, this formula for h implies that $0 < b \leq \sqrt{1200}$.

The volume is $V = b^2 h = b^2(1200 - b^2)/4b = 300b - b^3/4 \Rightarrow V'(b) = 300 - \frac{3}{4}b^2$.

$V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20$. Since $V'(b) > 0$ for $0 < b < 20$ and $V'(b) < 0$ for $b > 20$, there is an absolute maximum when $b = 20$ by the First Derivative Test for Absolute Extreme Values (see page 302).

If $b = 20$, then $h = (1200 - 20^2)/(4 \cdot 20) = 10$, so the largest possible volume is $b^2 h = (20)^2(10) = 4000 \text{ cm}^3$.



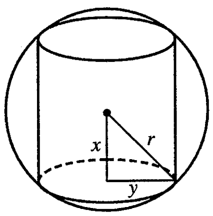
Notes on justifying the maximum: Although the above solution applies the First Derivative Test for Absolute Extrema, there are other approaches.

For instance, we could use the fact that on the "real-world" domain for $V(b)$, where $0 < b \leq \sqrt{1200}$, the value of $V''(b)$ is always negative (check this). Thus, the **Second Derivative Test for Absolute Extrema** can be applied, and the critical number $b = 20$ is a point of *absolute* maximum for $V(b)$.

Another acceptable approach applies the thinking of the **Closed Interval Method**, even though the domain $0 < b \leq \sqrt{1200}$ is not closed: the key is that $V(b)$ is still continuous on its domain, and its value *approaches* 0 as b approaches 0. So, you'd find the critical number $b = 20$ as above, and then simply compare $V(b)$ at $b = \sqrt{1200}$, $b = 20$, and $b \to 0$. Notice that the first and third values are 0, so the absolute maximum on this domain is at $b = 20$.

Solution to problem 19:

19.



The cylinder has volume $V = \pi y^2(2x)$. Also $x^2 + y^2 = r^2 \quad \Rightarrow \quad y^2 = r^2 - x^2$, so

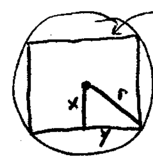
$V(x) = \pi(r^2 - x^2)(2x) = 2\pi(r^2x - x^3)$, where $0 \le x \le r$.

$V'(x) = 2\pi(r^2 - 3x^2) = 0 \quad \Rightarrow \quad x = r/\sqrt{3}$. Now $V(0) = V(r) = 0$, so there is a maximum when $x = r/\sqrt{3}$ and $V(r/\sqrt{3}) = \pi(r^2 - r^2/3)(2r/\sqrt{3}) = 4\pi r^3/(3\sqrt{3})$.

(absolute)

Closed Interval Method

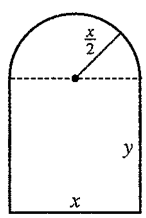
Solution tip: The key picture to draw is the one with the cylinder's top and bottom hidden from view:



top (so y is radius of cylinder's top)

Solution to problem 23:

23.



Perimeter $= 30 \quad \Rightarrow \quad 2y + x + \pi\left(\frac{x}{2}\right) = 30 \quad \Rightarrow$

$y = \frac{1}{2}\left(30 - x - \frac{\pi x}{2}\right) = 15 - \frac{x}{2} - \frac{\pi x}{4}$. The area is the area of the rectangle plus the area of the semicircle, or $xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2$, so $A(x) = x\left(15 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{2}x^2 - \frac{\pi}{8}x^2$.

$A'(x) = 15 - \left(1 + \frac{\pi}{4}\right)x = 0 \quad \Rightarrow \quad x = \frac{15}{1 + \pi/4} = \frac{60}{4 + \pi}$. Since $A''(x) = -\left(1 + \frac{\pi}{4}\right) < 0$ for all x, there is an absolute maximum at $x = \frac{60}{4+\pi}$, by the Second Derivative Test for Absolute Extrema.

The dimensions are $x = \frac{60}{4 + \pi}$ ft and $y = 15 - \frac{30}{4 + \pi} - \frac{15\pi}{4 + \pi} = \frac{60 + 15\pi - 30 - 15\pi}{4 + \pi} = \frac{30}{4 + \pi}$ ft, so the height of the rectangle is half the base.

Notes on justifying the maximum: The argument used above is that since since $A''(x) < 0$ for all x, the function $A(x)$ is concave down for *all* x; and this means that its single critical point is automatically a maximum — *both local and absolute.* This is the **Second Derivative Test for Absolute Extrema**. (See also Example 1 on pages 300-01, next-to-last paragraph, for another instance of this justification technique.)

Alternate justification approach: You could instead determine the domain of $A(x)$, which will be some closed interval. (For example, $x \ge 0$; also, the requirement that $y \ge 0$ will lead to an upper bound for x.) Then you could use the **Closed Interval Method**. However, this approach turns out to be somewhat messier in this case.