Math 41 – Fall 2006 – Final Exam Solutions

1. (15 points) Evaluate each of the following limits, showing all reasoning.

(a) $\lim_{h\to 0} \frac{e^{5+2h}-e^5}{h}$ Looks like $\frac{0}{0}$, so we try L'Hôpital:

$$\overset{L}{=} \lim_{h\to 0} \frac{e^{5+2h}\cdot 2}{1} = \boxed{2e^5}.$$

(**Alternative**: pull out a factor of e^5, we obtain $e^5 \cdot \lim_{h\to 0} \frac{e^{2h}-1}{h}$. The latter limit is the definition of the derivative of $f(x)=e^{2x}$ at $x=0$, so equals $f'(0)=2e^0=2$!)

(b) $\lim_{x\to 2^+} \frac{x^2-1}{x^2-5x+6}$ Numerator approaches 3 (nonzero) while denominator approaches $2^2-5\cdot 2+6=0$, and so we know the quotient approaches $\pm\infty$.

To determine sign of infinity, note that $x^2-5x+6=(x-2)(x-3)$, and for $x\to 2^+$ (i.e. $x>2$ and near 2), $x-2>0$ but $x-3<0$.

Thus $x^2-5x+6<0$ for $x\to 2^+$, and so the quotient has sign $\frac{\oplus}{\ominus}=\ominus$.

Thus, $\lim_{x\to 2^+} \frac{x^2-1}{x^2-5x+6} = \boxed{-\infty}$.

(c) $\lim_{x\to 0} \frac{\int_0^x \sin 2t\,dt}{\int_0^x \tan t\,dt}$

Since $\int_0^0 f(t)dt = 0$ for any f, this limit looks like $\frac{0}{0}$, and thus we try L'Hôpital.

$$\lim_{x\to 0} \frac{\int_0^x \sin 2t\,dt}{\int_0^x \tan t\,dt} = \lim_{x\to 0} \frac{\frac{d}{dx}\int_0^x \sin 2t\,dt}{\frac{d}{dx}\int_0^x \tan t\,dt} = \lim_{x\to 0} \frac{\sin 2x}{\tan x} \quad \text{(by FTC)}$$

Still looks like $\frac{0}{0}$, so we try L'Hôpital again.

$$\rightarrow \quad = \lim_{x\to 0} \frac{2\cos 2x}{\sec^2 x} = \frac{2\cos 0}{\sec^2 0} = \boxed{2}.$$

2. (15 points) Differentiate, using any method you choose. You do not have to simplify your answers.

(a) $y = (\cos x)^x$

$\hookrightarrow \quad \ln y = \ln((\cos x)^x) = x \ln(\cos x)$, so

$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{d}{dx}(x \ln(\cos x)) = \ln(\cos x) + x \cdot \frac{1}{\cos x} \cdot (-\sin x)$, so

$\frac{dy}{dx} = y \cdot \left(\ln(\cos x) - x \cdot \frac{\sin x}{\cos x}\right) = \boxed{(\cos x)^x \cdot \left(\ln(\cos x) - x \cdot \frac{\sin x}{\cos x}\right)}$.

(b) $f(z) = e^{\sec^2 z} \cdot \ln z$

$f'(z) = \frac{d}{dz}(e^{\sec^2 z}) \cdot \ln z + \frac{1}{z} \cdot e^{\sec^2 z}$

$= \boxed{e^{\sec^2 z} \cdot (2\sec z)(\sec z \tan z) \cdot \ln z + \frac{1}{z} \cdot e^{\sec^2 z}}$

(c) $g(x) = \int_1^{\sin x} (t^2 + 1)^{2006}\, dt$

If $F(t)$ is an antiderivative of $(t^2+1)^{2006}$, then $g(x) = F(t)\Big]_{t=1}^{t=\sin x}$

$= F(\sin x) - F(1)$.

Thus, $g'(x) = \frac{d}{dx}(F(\sin x)) - \frac{d}{dx}(F(1))$

$= F'(\sin x) \cdot \cos x - 0 = \boxed{(\sin^2 x + 1)^{2006} \cdot \cos x}$.

(Alternatively: Let $u = \sin x$; then $g(x) = \int_1^{u(x)} (t^2+1)^{2006}\, dt$, so that

$g'(x) = \frac{dg}{dx} = \frac{dg}{du} \cdot \frac{du}{dx} = \left(\frac{d}{du} \int_1^u (t^2+1)^{2006}\, dt\right) \cdot \left(\frac{d}{dx}(\sin x)\right) = (u^2+1)^{2006} \cdot \cos x$

$= \underline{(\sin^2 x+1)^{2006} \cdot \cos x}$.)

3. (24 points) Evaluate each of the following integrals, showing all reasoning.

(a) $\displaystyle\int_2^3 \frac{3x^2+2x+1}{x}\,dx \quad = \quad \int_2^3 \left(3x+2+\frac{1}{x}\right)dx$

$$= \left.\frac{3}{2}x^2 + 2x + \ln|x|\right]_{x=2}^{x=3}$$

$$= \left(\frac{3\cdot 3^2}{2} + 2\cdot 3 + \ln 3\right) - \left(\frac{3\cdot 2^2}{2} + 2\cdot 2 + \ln 2\right)$$

(b) $\displaystyle\int \tan x\,dx \quad = \int \frac{\sin x}{\cos x}\,dx.$ (Definition of $\tan x$)

Let $\left\{\begin{array}{l} u = \cos x \\ du = -\sin x\,dx \end{array}\right\} \Rightarrow \displaystyle\int \frac{\sin x}{\cos x}\,dx = \int \frac{-du}{u}$

$$= -\ln|u| + C$$

$$= \boxed{-\ln|\cos x| + C}.$$

(c) $\int_{-1}^{1} xe^x\,dx$ $\left\{ \text{Parts: } \begin{array}{ll} f = x & f' = 1 \\ g' = e^x & g = e^x \end{array} \right\}$

$$\Rightarrow \quad = xe^x\Big]_{-1}^{1} - \int_{-1}^{1} e^x dx = xe^x\Big]_{-1}^{1} - e^x\Big]_{-1}^{1}$$

$$= (e) - (-e^{-1}) - (e - e^{-1})$$

$$= e + \frac{1}{e} - e + \frac{1}{e} = \boxed{\frac{2}{e}}.$$

(d) $\int x\sqrt{2x-1}\,dx$

Let $\left\{ \begin{array}{l} u = 2x-1 \\ du = 2dx \end{array} \right\} \Rightarrow \int x\sqrt{2x-1}\,dx = \int \left(\frac{u+1}{2}\right) \cdot \sqrt{u} \cdot \left(\frac{du}{2}\right)$

$$= \frac{1}{4}\int (u+1)\sqrt{u}\,du$$

$$= \frac{1}{4}\int u^{3/2} + u^{1/2}\,du$$

$$= \frac{1}{4}\left(\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right) + C$$

$$= \boxed{\frac{(2x-1)^{5/2}}{10} + \frac{(2x-1)^{3/2}}{6} + C}$$

(Can also be done using integration by parts, and a slightly different answer (but equivalent one) is obtained.)

(e) $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$

Let $\left\{\begin{array}{l} u=\sqrt{x} \\ du=\frac{1}{2}\cdot\frac{1}{\sqrt{x}}dx \end{array}\right\} \Rightarrow \displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}dx = \int \arctan u \cdot 2du$

$$= 2\int \arctan u\, du.$$

By parts: $\left\{\begin{array}{ll} f=\arctan u & f'=\frac{1}{1+u^2} \\ g'=1 & g=u \end{array}\right\} \Rightarrow$

$$= 2\left[u\arctan u - \int \frac{u}{1+u^2}du\right]$$

$$= 2u\arctan u - 2\int \frac{u}{1+u^2}du$$

Let $\left\{\begin{array}{l} w=1+u^2 \\ dw=2u\,du \end{array}\right\} \Rightarrow$

$$= 2u\arctan u - \int \frac{dw}{w}$$

$$= 2u\arctan u - \ln|w| + C$$

$$= 2u\arctan u - \ln|1+u^2| + C$$

$$= 2\sqrt{x}\arctan\sqrt{x} - \ln|1+(\sqrt{x})^2| + C$$

$$= \boxed{2\sqrt{x}\arctan\sqrt{x} - \ln(1+x) + C}$$

(Note: okay to drop absolute value bars, since it's understood here that $x>0$ (otherwise $\sqrt{x}$ would not be defined).)

4. (8 points) The rate at which the world's oil is being consumed, measured in billions of barrels per year, is given by the function $r(t)$, where t is measured in years and $t = 0$ represents January 1, 2000:

$$r(t) = 32e^{0.05t}$$

(a) Calculate $\int_0^6 r(t)\,dt$. (There is no need to simplify your answer.)

$$\int_0^6 r(t)dt = \int_0^6 32e^{0.05t}dt = 32\int_0^6 e^{0.05t}dt$$

$$\left\{\begin{array}{l} u = 0.05t \\ du = 0.05 \cdot dt \end{array}\right\} \Rightarrow \quad = 32\int_0^{6\cdot 0.05} e^u \cdot \frac{du}{0.05}$$

$$= \boxed{\frac{32}{0.05}\left[e^{6\cdot 0.05} - e^0\right]}$$

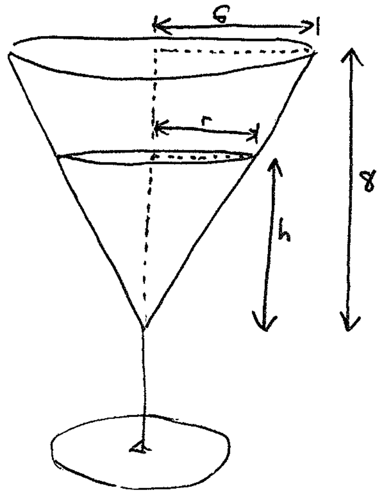
(b) What are the units of your answer to part (a)? Write a sentence interpreting the quantity you found as something having to do with oil.

Units of $\int_0^6 r(t)dt$ are: billions of barrels (of oil).

The quantity $\int_0^6 r(t)dt$ is the net (or total) amount of oil consumed between January 1, 2000 and January 1, 2006, in billions of barrels.

(This is because the function $r(t)$ may be thought of as the instantaneous rate of change, or derivative, of the total amount of oil consumed up to time t; and the "Net Change Theorem" says that $\int_0^6 r(t)dt$ is therefore the net change in the total amount consumed between times 1/1/2000 and 1/1/2006, i.e., the net amount consumed during this six-year period.)

5. (8 points) A cocktail glass has a cone-shaped bowl that contains a tropical drink. The drink is being sipped through a straw at the rate of 5 cm^3/min. If the cone is 8 cm tall with a radius of 6 cm at the top, how quickly is the level of liquid dropping when the level is 4 cm? (You should ignore any role played by the straw's negligible volume.)



Let r = radius at the top of the remaining liquid, and h = height of the remaining liquid, at any given moment.

By similar triangles, $\frac{r}{h} = \frac{6}{8}$, so

$$r = \frac{3}{4}h.$$

At any given moment, we have that the volume V of remaining liquid satisfies:

- $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h$ and
- $\frac{dV}{dt} = -5$ cm^3/min.

Thus, $$\frac{dV}{dt} = \frac{d}{dt}\left(\frac{1}{3}\pi \cdot \frac{9}{16}h^3\right) = \frac{1}{3}\pi \cdot \frac{9}{16} \cdot \frac{d}{dt}(h^3)$$

$$= \frac{1}{3}\pi \cdot \frac{9}{16} \cdot 3h^2 \frac{dh}{dt} = -5,$$

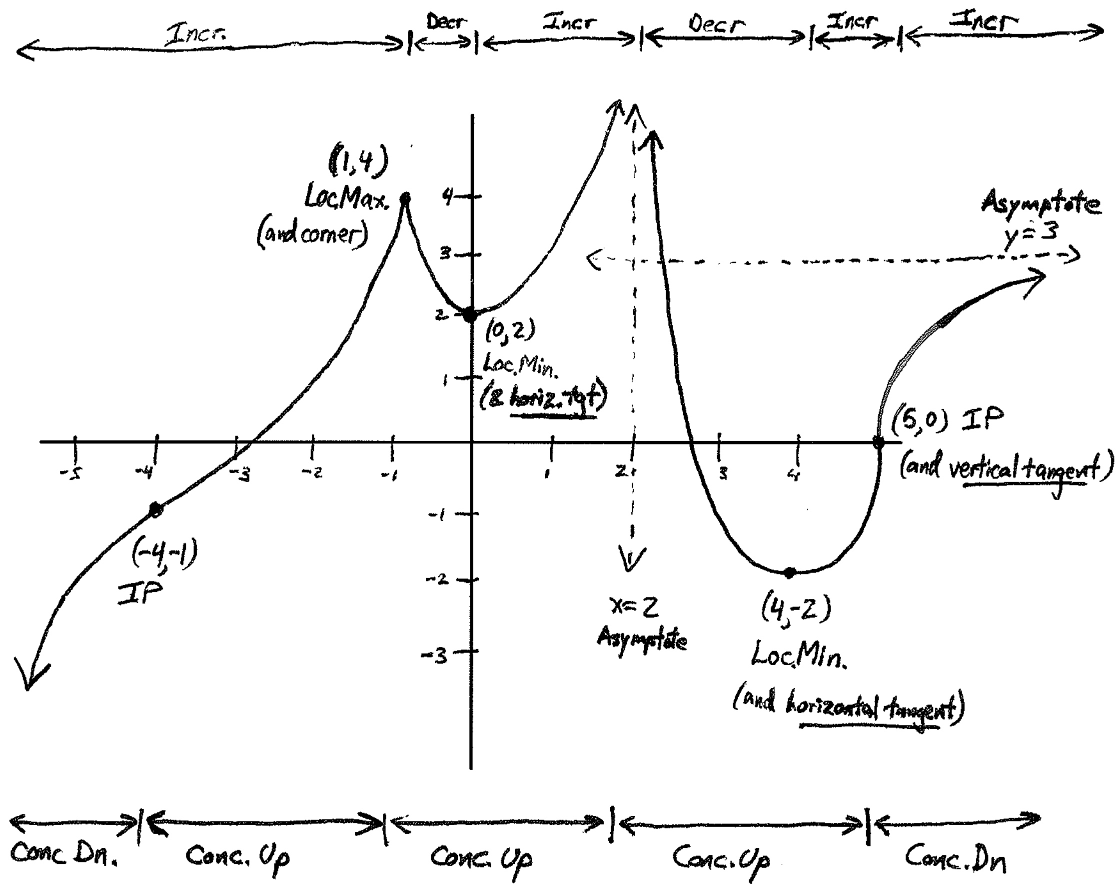
which implies $$\frac{dh}{dt} = \frac{-5}{\frac{1}{3}\pi \cdot \frac{9}{16} \cdot 3h^2} = \frac{-5 \cdot 16}{9\pi h^2}.$$

At the moment when $h = 4$ cm, we have $\frac{dh}{dt} = -\frac{5 \cdot 16}{9\pi \cdot 16} = \boxed{-\frac{5}{9\pi} \frac{\text{cm}}{\text{min}}}$.

6. (15 points) Sketch the graph of a function f with all of the following properties:

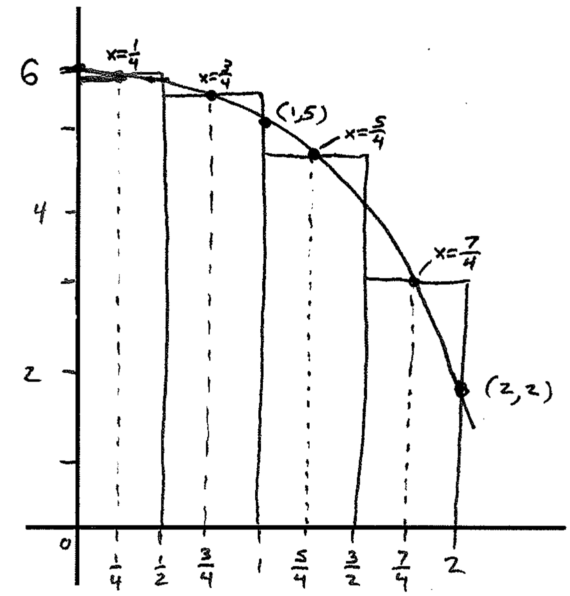
- $f(x)$ is continuous on its entire domain, which is all x except $x = 2$.
- $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = 3$.
- $\lim_{x \to 2} f(x) = \infty$.
- $f'(x)$ is continuous at all x except $x = -1$, $x = 2$, and $x = 5$.
- $f'(x) > 0$ for $x < -1$ and for $0 < x < 2$ and for $4 < x < 5$ and for $x > 5$.
- $f'(x) < 0$ for $-1 < x < 0$ and for $2 < x < 4$.
- $\lim_{x \to -1^-} f'(x) = 3$ and $\lim_{x \to -1^+} f'(x) = -3$.
- $\lim_{x \to 5} f'(x) = \infty$.
- $f''(x) > 0$ for $-4 < x < -1$ and for $-1 < x < 2$ and for $2 < x < 5$.
- $f''(x) < 0$ for $x < -4$ and for $x > 5$.
- $f(-4) = -1$, $f(-1) = 4$, $f(0) = 2$, $f(4) = -2$, and $f(5) = 0$.

Label all horizontal and vertical asymptotes, local extrema, and inflection points.



7. (15 points) Let $f(x) = 6 - x^2$.

(a) On the axes below, sketch a graph of f over the domain $[0, 2]$, and then draw the approximating rectangles that are used to estimate the area under the curve (and above the y-axis) between $x = 0$ and $x = 2$ according to the *Midpoint Rule*; use $n = 4$ rectangles.



$(x_0 \ \bar{x}_1 \ x_1 \ \bar{x}_2 \ x_2 \ \bar{x}_3 \ x_3 \ \bar{x}_4 \ x_4)$ – labels not necessary.

(b) Write an expression involving only numbers that represents the area estimate using these rectangles. (You do *not* have to expand or simplify the expression!)

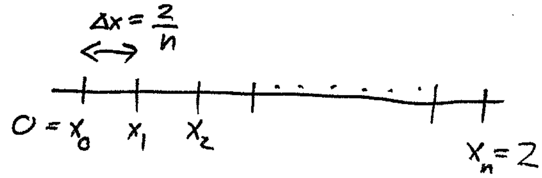
$$\text{Area} = \text{height}_1 \cdot \text{width} + \text{height}_2 \cdot \text{width} + \text{height}_3 \cdot \text{width} + \text{height}_4 \cdot \text{width}$$

$$= \text{width} \cdot (\text{height}_1 + \text{height}_2 + \text{height}_3 + \text{height}_4)$$

$$= \frac{1}{2} \cdot \left(f\left(\tfrac{1}{4}\right) + f\left(\tfrac{3}{4}\right) + f\left(\tfrac{5}{4}\right) + f\left(\tfrac{7}{4}\right) \right)$$

$$= \frac{1}{2} \cdot \left(6 - \left(\tfrac{1}{4}\right)^2 + 6 - \left(\tfrac{3}{4}\right)^2 + 6 - \left(\tfrac{5}{4}\right)^2 + 6 - \left(\tfrac{7}{4}\right)^2 \right)$$

(c) Find the exact area of the same region by evaluating the limit of a Riemann sum that uses the *Right Endpoint Rule.* (That is, do not use the Fundamental Theorem of Calculus.) Show all reasoning.



Sample points $x_i^* = x_i = \frac{2i}{n}$ for Right Endpoint Rule.

$$\text{Area} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f\left(\frac{2i}{n}\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(6-\left(\frac{2i}{n}\right)^2\right)\cdot\frac{2}{n}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{12}{n} - \frac{8i^2}{n^3}\right)$$

$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{12}{n} - \frac{8}{n^3}\sum_{i=1}^{n} i^2\right]$$

using reference formula

$$= \lim_{n\to\infty} \left[\frac{12}{n}\cdot n - \frac{8}{n^3}\cdot\frac{n(n+1)(2n+1)}{6}\right]$$

$$= 12 - \lim_{n\to\infty} \frac{8n(n+1)(2n+1)}{6n^3} = 12 - \lim_{n\to\infty} \frac{8}{6}\cdot\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

$$= 12 - \frac{8}{6}\cdot 1\cdot 2$$

$$= 12 - \frac{8}{3} = \boxed{\frac{28}{3}}$$

8. (7 points) Put the following quantities in increasing order (from smallest number to largest). You do not need to justify your answer.

Final Answer: F < D < G < B < E < A < C.

A • $\int_2^6 \ln t \, dt$

B • $\ln 2 + \ln 3 + \ln 4 + \ln 5$

C • $\ln 3 + \ln 4 + \ln 5 + \ln 6$

D • The number 0

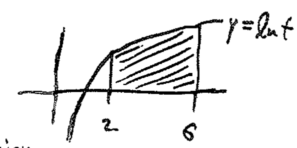
E • $\displaystyle\sum_{i=0}^{7} \frac{\ln\left(2+\frac{i}{2}\right)}{2}$

F • $\ln(2/6)$ ← Is negative, because $\frac{2}{6} < 1$.

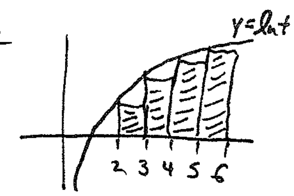
G • $\displaystyle\lim_{h\to 0} \frac{\ln(2+h) - \ln 2}{h}$ ← Can be computed to be $\frac{1}{2}$, either using L'Hôpital's rule (since $\frac{0}{0}$) or by noting this is the limit definition of the derivative of $\ln x$ at $x=2$. (Thus, $\frac{d}{dx}(\ln x) = \frac{1}{x}$, so equals $\frac{1}{2}$ at $x=2$.)

Quantities A, B, C, and E are all able to be interpreted as areas. (All are greater than $\frac{1}{2}$ in size.)

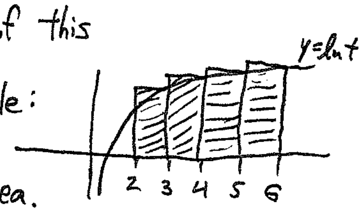
- $\int_2^6 \ln t \, dt$ is the area under $y = \ln t$ between $t=2$ and $t=6$: The other three quantities represent approximations of the area of this region.



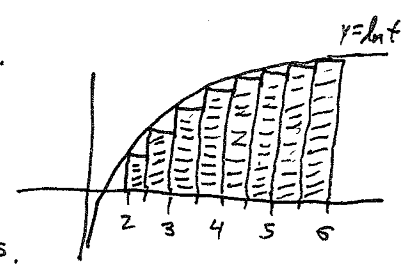
- $\ln 2 + \ln 3 + \ln 4 + \ln 5$ is a Riemann Sum approximation of the area of the same region, using 4 rectangles of width 1, and the Left Endpoint Rule: Since $y = \ln t$ is increasing, this is an underestimate of the actual area.



- $\ln 3 + \ln 4 + \ln 5 + \ln 6$ is a Riemann Sum approximation of the area of this region, using 4 rectangles of width 1, and the Right Endpt Rule: Since $y = \ln t$ is increasing, this is an overestimate of the actual area.



- $\displaystyle\sum_{i=0}^{7} \frac{\ln\left(2+\frac{i}{2}\right)}{2} = \frac{\ln 2}{2} + \frac{\ln\left(\frac{5}{2}\right)}{2} + \frac{\ln(3)}{2} + \dots + \frac{\ln\left(\frac{11}{2}\right)}{2}$ is a Riemann Sum approx. of the area of the same region, using 8 rectangles of width $\frac{1}{2}$, and the Left Endpt Rule. It is still an underestimate of the actual area, but is a better approx. than using 4 rectangles.



9. (10 points)

(a) Verify the following indefinite integral expression by differentiating.

$$\int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$

$$\frac{d}{dx}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C\right) = \frac{1}{2}\sqrt{1-x^2} + \frac{x}{2}\cdot\frac{1}{2}(1-x^2)^{-1/2}\cdot(-2x) + \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}$$

$$= \frac{1}{2}\sqrt{1-x^2} + \frac{-x^2}{2}\cdot\frac{1}{\sqrt{1-x^2}} + \frac{1}{2\sqrt{1-x^2}}$$

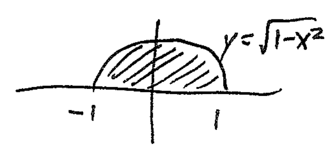
$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot\left((1-x^2) + (-x^2) + 1\right)$$

$$= \frac{1}{2}\cdot\frac{1}{\sqrt{1-x^2}}\cdot(2-2x^2)$$

$$= \frac{1}{2}\cdot(1-x^2)^{-1/2}\cdot 2\cdot(1-x^2) = \boxed{\sqrt{1-x^2}}$$ as desired.

(b) Use the *above formula* to compute the area of a *semicircle* of radius 1, centered at the origin. Show all steps in your calculation.

A semicircle of radius 1 centered at the origin has boundary curve given by $y = \sqrt{1-x^2}$ (from $x^2+y^2=1$).



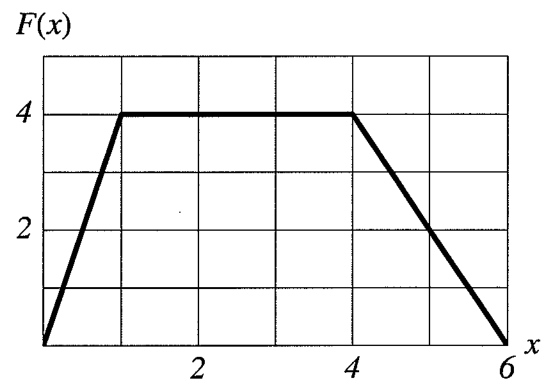
Thus, area $= \displaystyle\int_{-1}^{1}\sqrt{1-x^2}\,dx = \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x\right]_{x=-1}^{x=1}$

$$= \left(\frac{1}{2}\cdot\sqrt{1-1} + \frac{1}{2}\arcsin 1\right) - \left(\frac{-1}{2}\sqrt{1-1} + \frac{1}{2}\arcsin(-1)\right)$$

$$= \frac{1}{2}\arcsin 1 - \frac{1}{2}\arcsin(-1) = \frac{1}{2}\cdot\frac{\pi}{2} - \frac{1}{2}\cdot\left(-\frac{\pi}{2}\right) = 2\cdot\frac{\pi}{4} = \boxed{\frac{\pi}{2}}.$$

$\left(\sin\frac{\pi}{2} = 1, \text{ so } \arcsin 1 = \frac{\pi}{2}.\right)$

10. (5 points) Find a function $g(x)$ such that the graph of $F(x) = \int_0^x g(t)\,dt$ is the graph below.



(Specify such a $g(x)$ on the domain [0,6].)

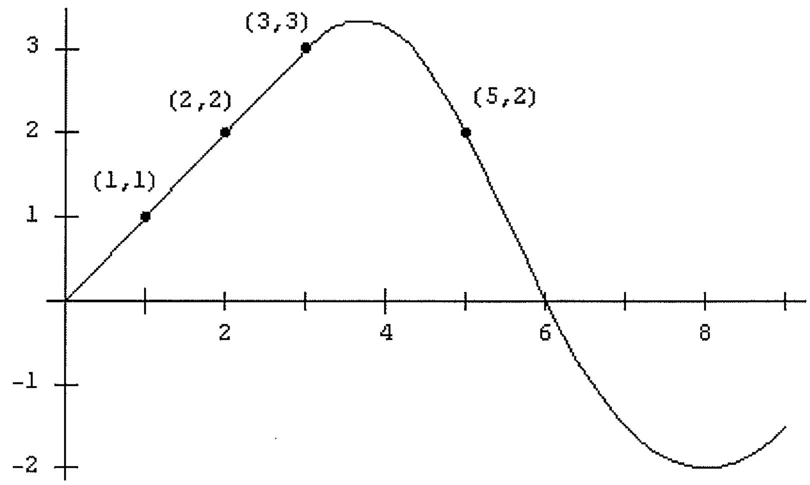
Since $F'(x) = g(x)$, we must have the following values for g:

$$g(x) = \begin{cases} 4 & 0 < x < 1 \\ 0 & 1 < x < 4 \\ -2 & 4 < x < 6 \end{cases}$$

(Values of g at $x=0$, $x=1$, $x=4$, $x=6$ are not uniquely determined by the graph of F, since g will be discontinuous -- and certainly F is non-differentiable -- at these points.)

11. (15 points) Let $s(t)$ be the position, in meters, at time t seconds of a particle moving along a coordinate axis, and *suppose the position at time 0 is 1 m* (i.e., $s(0) = 1$ m).

As usual, we write $v(t)$ for the velocity function (in meters per second). Suppose the graph of the *velocity function* $v(t)$ is shown below:



Use the information above to answer the following questions. *Give reasons for your answers.*

(a) What is the particle's velocity at time $t = 5$?

By the graph above, $v(5) = 2$ m/s.

(b) Is the acceleration of the particle at time $t = 5$ positive or negative?

Acceleration $a(t) = v'(t)$, so acceleration is given by the slope of the above curve. Thus, at $t=5$, acceleration is negative (v is decreasing at 5).

(c) What is the particle's position at time $t = 3$?

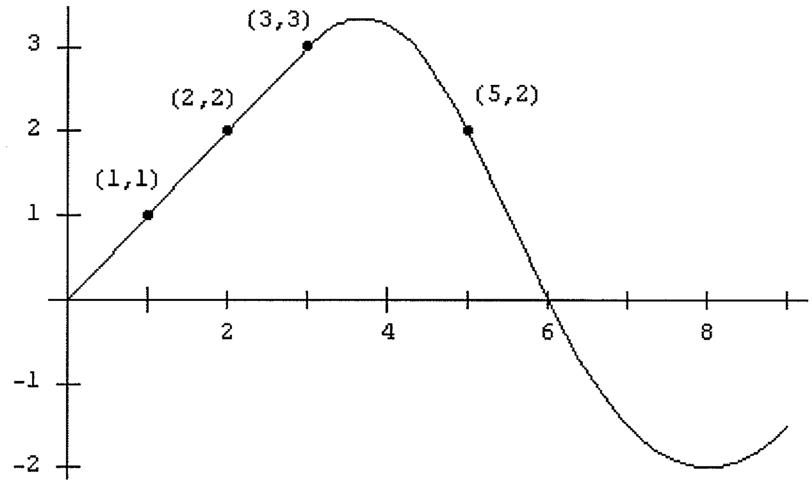
By the net change theorem, $\begin{pmatrix}\text{change in position} \\ \text{between } t=0 \ \& \ t=3\end{pmatrix} = \int_0^3 v(t)\,dt$

$= \begin{matrix}\text{Area under graph above} \\ \text{between } t=0 \ \& \ t=3\end{matrix} = \frac{1}{2}(3)(3) = \frac{9}{2}$,

so position at $t=3$ is given by $s(3) = s(0) + (\text{change in pos})$

$= 1 + \frac{9}{2} = \boxed{\frac{11}{2} \text{ m}}$.

For easy reference, here again is the graph of the *velocity function*:



(d) At what time during the first 9 seconds does s have its largest value?

Using same reasoning as in (c), we have $s(T) = s(0) + \int_0^T v(t)\,dt$.

So s is largest when $\int_0^T v(t)\,dt$ is largest, i.e. when area under above graph betw. 0 & T largest. This occurs at $\boxed{T=6}$ since for $T>6$ the area is reduced by the amount under the axis.

(e) Approximately when is the acceleration zero?

Since $a(t) = v'(t)$, we cite the values of t where graph above is horizontal: approx $t = 3\frac{3}{4}$ and $t = 8$ sec.

(f) When is the particle moving toward the origin? Away from the origin?

Position starts as positive at $t=0$, and between $t=0$ & $t=6$ the velocity is positive, so position increases, meaning motion is away from origin.

After $t=6$, velocity is negative, meaning motion is toward origin (since position is positive, and according to part g, position never is negative).

(g) On which side (positive or negative) of the origin does the particle lie at time $t = 9$?

Position is positive: since $s(T) = s(0) + \int_0^T v(t)\,dt = 1 + \int_0^T v(t)\,dt$, position at time $t=T$ can only be negative if $\int_0^T v(t)\,dt$ is less than -1 for some T. But as noted, this area starts positive for small T, and although is reduced after $T=6$, the total area between $T=6$ and $T=9$ does not sufficiently reduce the positive area between 0 and 6.

(h) Given that $\int_0^6 v(t)\,dt = 11.5$ and $\int_6^9 v(t)\,dt = -4.5$, find the *total distance traveled* by the particle in the first 9 seconds.

$$\text{Total distance traveled} = \int_0^9 |v(t)|\,dt$$

$$= \int_0^6 v(t)\,dt + \left|\int_6^9 v(t)\,dt\right| \quad \left(\begin{array}{l}\text{because } v \geq 0 \text{ on } [0,6],\\ \text{and } v \leq 0 \text{ on } [6,9]\end{array}\right)$$

$$= 11.5 + 4.5 = \boxed{16\text{ m}}.$$

12. (8 points) Suppose that we don't have a formula for $h(x)$, but we know that

$$h(3) = -6 \quad \text{and} \quad h'(x) = \sqrt[3]{17 - x^2} \text{ for all } x.$$

(a) Use a linear approximation to estimate $h(2.99)$ and $h(3.02)$.

We use the linear approximation at $x=3$: since $h(3) = -6$ and $h'(3) = \sqrt[3]{17-9} = 2$,

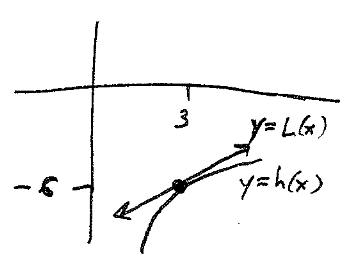
the linearization of h at 3 is $L(x) = -6 + 2(x-3)$.

Thus, $h(2.99) \approx L(2.99) = -6 + 2(2.99-3) = -6 + 2(-0.01) = -6.02$.

and $h(3.02) \approx L(3.02) = -6 + 2(3.02-3) = -6 + 2(0.02) = -5.96$.

(b) Are your estimates in part (a) too large or too small? Explain.

Since $h''(x) = \frac{1}{3}(17-x^2)^{-2/3}(-2x) < 0$ near $x=3$, the graph of h near $x=3$ is concave down. Thus, the linear approximation, which gives the y-coordinate of the tangent line at $x=3$, will be <u>too large</u> because the tangent line lies above a concave-down curve.



13. (5 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

T **F** If c is a critical number of a function f and also $f''(c) = 0$, then by the Second Derivative Test, it follows that f achieves neither a local maximum nor a local minimum at $x = c$.

$f''(c) = 0$ means test is inconclusive.

T **F** The absolute maximum value of a continuous function $f(x)$ defined on a closed interval $[a, b]$ can only be realized at an endpoint ($x = a$ or $x = b$) or at a point where the graph of f has a horizontal tangent.

Another potential location for abs. max: any place where f is non-differentiable.

T **F** If h is continuous, decreasing, and concave down for all x, then $h(x)$ must be negative for some sufficiently large value of x.

A variant of the Mean Value Theorem can be used to prove this rigorously, but draw a picture of this situation to convince yourself on an intuitive level.

T **F** A recommended initial "guess" when using Newton's method to solve the equation $f(x) = 0$ is an x_1 such that $f'(x_1)$ is very close to zero.

Bad idea: if $f'(x_1)$ nearly zero, then the tangent line will have an x-intercept that is quite far from $x = x_1$.

T **F** If g is an even function that is continuous at all values, then $\int_{-a}^{a} xg(x)\,dx = 0$ for any value of a.

$x \cdot g(x)$ is odd, since $(-x) \cdot g(-x) = (-x) \cdot g(x) = -x \cdot g(x)$,

and the integral of any odd function over an interval $[-a, a]$ is zero.