Math 41: Calculus
Final Exam — December 11, 2006

Name : ____

Section Leader (Circle one) : Chang Ivanov Mathews Requeijo Segerman

Section Time (Circle one): 11:00 1:15

- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, please show all of your work and justify your answer. You do not need to simplify your answers unless specifically instructed to do so.
- If you need extra room, use the back sides of each page. If you must use extra paper, make sure to write your name on it and attach it to this exam. Do not unstaple or detach pages from this exam.
- Please sign the following:

"On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination."

Signature: ___________________________

The following boxes are strictly for grading purposes. Please do not mark.

1	15		**8**	7	
2	15		**9**	10	
3	24		**10**	5	
4	8		**11**	15	
5	8		**12**	8	
6	15		**13**	5	
7	15		**Total**	150	

GOOD LUCK ON THE REST OF YOUR EXAMS, AND HAVE A GREAT WINTER BREAK!

1. (15 points) Evaluate each of the following limits, showing all reasoning.

   (a) $\lim_{h \to 0} \dfrac{e^{5+2h} - e^5}{h}$

   (b) $\lim_{x \to 2^+} \dfrac{x^2 - 1}{x^2 - 5x + 6}$

(c) $\lim_{x \to 0} \frac{\int_0^x \sin 2t \, dt}{\int_0^x \tan t \, dt}$

2. (15 points) Differentiate, using any method you choose. You do not have to simplify your answers.

   (a) $y = (\cos x)^x$

   (b) $f(z) = e^{\sec^2 z} \cdot \ln z$

   (c) $g(x) = \displaystyle\int_1^{\sin x} (t^2 + 1)^{2006}\, dt$

3. (24 points) Evaluate each of the following integrals, showing all reasoning.

(a) $\displaystyle\int_2^3 \frac{3x^2 + 2x + 1}{x}\,dx$

(b) $\displaystyle\int \tan x\,dx$

(c) $\int_{-1}^{1} xe^x \, dx$

(d) $\int x\sqrt{2x-1}\, dx$

(e) $\int \frac{\arctan \sqrt{x}}{\sqrt{x}}\, dx$

4. (8 points) The rate at which the world's oil is being consumed, measured in billions of barrels per year, is given by the function $r(t)$, where t is measured in years and $t = 0$ represents January 1, 2000:

$$r(t) = 32e^{0.05t}$$

(a) Calculate $\int_0^6 r(t)\,dt$. (There is no need to simplify your answer.)

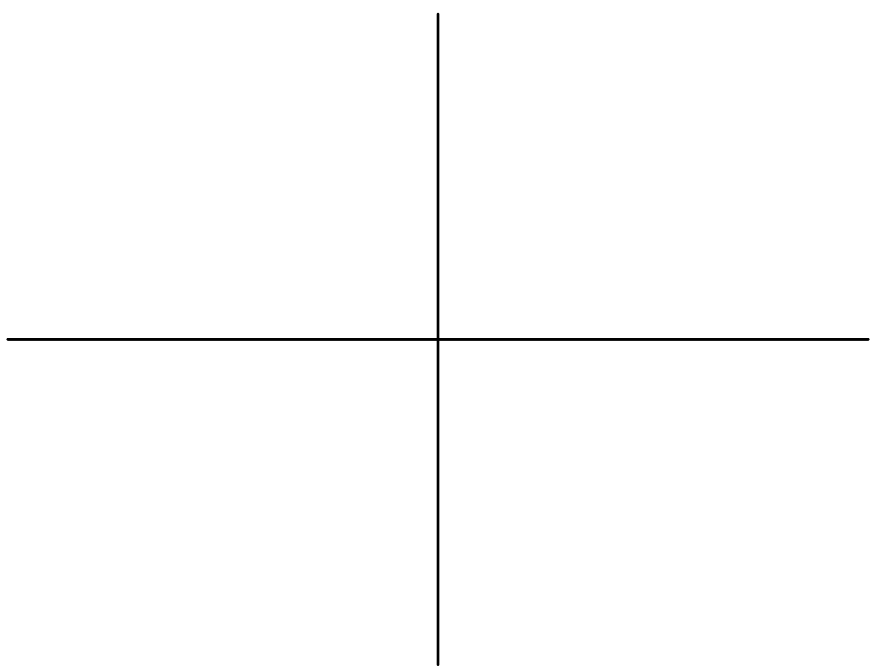
(b) What are the units of your answer to part (a)? Write a sentence interpreting the quantity you found as something having to do with oil.

5. (8 points) A cocktail glass has a cone-shaped bowl that contains a tropical drink. The drink is being sipped through a straw at the rate of 5 cm^3/min. If the cone is 8 cm tall with a radius of 6 cm at the top, how quickly is the level of liquid dropping when the level is 4 cm? (You should ignore any role played by the straw's negligible volume.)

6. (15 points) Sketch the graph of a function f with all of the following properties:

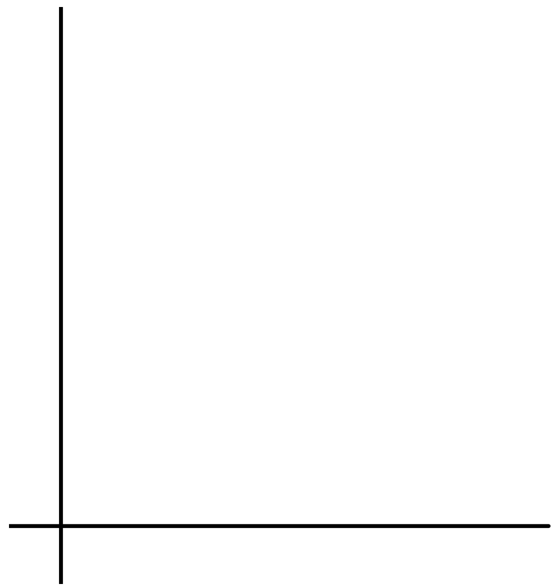
- $f(x)$ is continuous on its entire domain, which is all x except $x = 2$.
- $\lim_{x \to -\infty} f(x) = -\infty$ and $\lim_{x \to \infty} f(x) = 3$.
- $\lim_{x \to 2} f(x) = \infty$.
- $f'(x)$ is continuous at all x except $x = -1$, $x = 2$, and $x = 5$.
- $f'(x) > 0$ for $x < -1$ and for $0 < x < 2$ and for $4 < x < 5$ and for $x > 5$.
- $f'(x) < 0$ for $-1 < x < 0$ and for $2 < x < 4$.
- $\lim_{x \to -1^-} f'(x) = 3$ and $\lim_{x \to -1^+} f'(x) = -3$.
- $\lim_{x \to 5} f'(x) = \infty$.
- $f''(x) > 0$ for $-4 < x < -1$ and for $-1 < x < 2$ and for $2 < x < 5$.
- $f''(x) < 0$ for $x < -4$ and for $x > 5$.
- $f(-4) = -1$, $f(-1) = 4$, $f(0) = 2$, $f(4) = -2$, and $f(5) = 0$.

Label all horizontal and vertical asymptotes, local extrema, and inflection points.



7. (15 points) Let $f(x) = 6 - x^2$.

(a) On the axes below, sketch a graph of f over the domain $[0, 2]$, and then draw the approximating rectangles that are used to estimate the area under the curve (and above the y-axis) between $x = 0$ and $x = 2$ according to the *Midpoint Rule*; use $n = 4$ rectangles.



(b) Write an expression involving only numbers that represents the area estimate using these rectangles. (You do *not* have to expand or simplify the expression!)

(c) Find the exact area of the same region by evaluating the limit of a Riemann sum that uses the *Right Endpoint Rule*. (That is, do not use the Fundamental Theorem of Calculus.) Show all reasoning.

8. (7 points) Put the following quantities in increasing order (from smallest number to largest). You do not need to justify your answer.

- $\int_2^6 \ln t \, dt$
- $\ln 2 + \ln 3 + \ln 4 + \ln 5$
- $\ln 3 + \ln 4 + \ln 5 + \ln 6$
- The number 0
- $\displaystyle\sum_{i=0}^{7} \frac{\ln\left(2 + \frac{i}{2}\right)}{2}$
- $\ln(2/6)$
- $\displaystyle\lim_{h \to 0} \frac{\ln(2+h) - \ln 2}{h}$

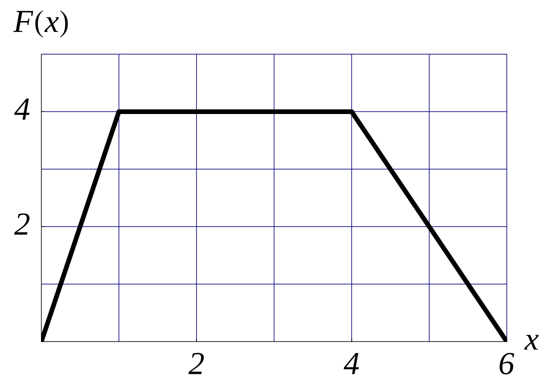
9. (10 points)

(a) Verify the following indefinite integral expression by differentiating.

$$\int \sqrt{1-x^2}\,dx = \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\arcsin x + C$$

(b) Use the *above formula* to compute the area of a *semicircle* of radius 1, centered at the origin. Show all steps in your calculation.

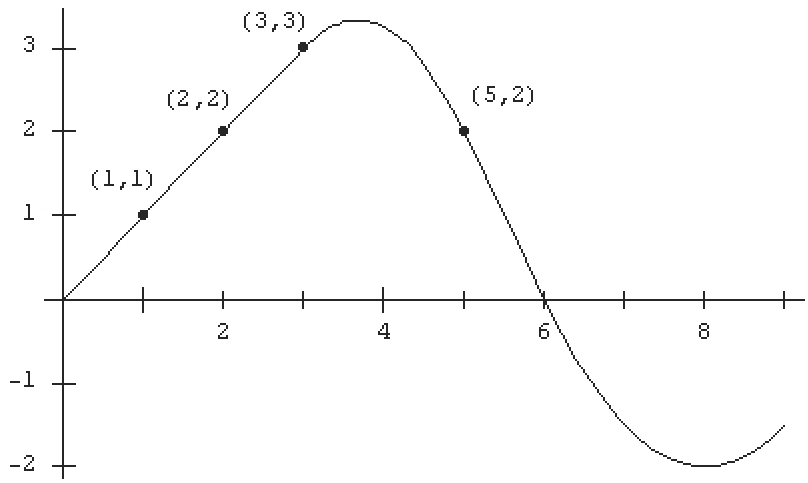
10. (5 points) Find a function $g(x)$ such that the graph of $F(x) = \int_0^x g(t)\,dt$ is the graph below.



(Specify such a $g(x)$ on the domain [0,6].)

11. (15 points) Let $s(t)$ be the position, in meters, at time t seconds of a particle moving along a coordinate axis, and *suppose the position at time 0 is 1 m* (i.e., $s(0) = 1$ m).

As usual, we write $v(t)$ for the velocity function (in meters per second). Suppose the graph of the *velocity function* $v(t)$ is shown below:



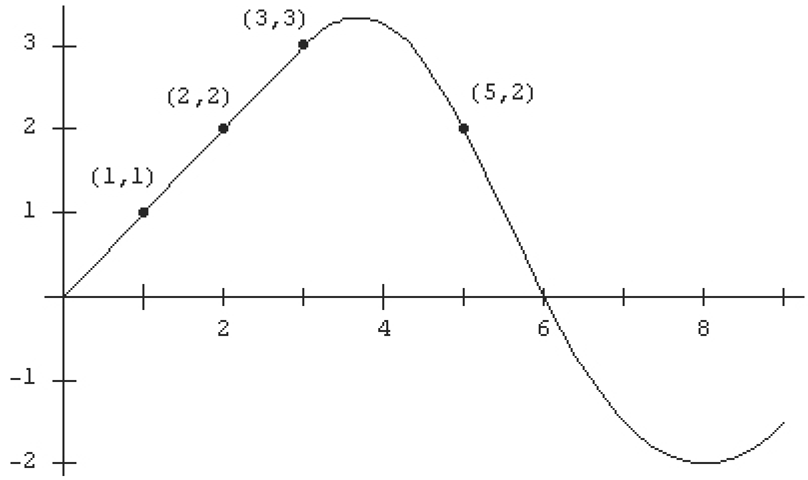
Use the information above to answer the following questions. *Give reasons for your answers.*

(a) What is the particle's velocity at time $t = 5$?

(b) Is the acceleration of the particle at time $t = 5$ positive or negative?

(c) What is the particle's position at time $t = 3$?

For easy reference, here again is the graph of the *velocity function*:



(d) At what time during the first 9 seconds does s have its largest value?

(e) Approximately when is the acceleration zero?

(f) When is the particle moving toward the origin? Away from the origin?

(g) On which side (positive or negative) of the origin does the particle lie at time $t = 9$?

(h) Given that $\int_0^6 v(t)\,dt = 11.5$ and $\int_6^9 v(t)\,dt = -4.5$, find the *total distance traveled* by the particle in the first 9 seconds.

12. (8 points) Suppose that we don't have a formula for $h(x)$, but we know that

$$h(3) = -6 \quad \text{and} \quad h'(x) = \sqrt[3]{17 - x^2} \text{ for all } x.$$

(a) Use a linear approximation to estimate $h(2.99)$ and $h(3.02)$.

(b) Are your estimates in part (a) too large or too small? Explain.

13. (5 points) Mark each statement below as *true* or *false* by circling **T** or **F**. No justification is necessary.

T **F** If c is a critical number of a function f and also $f''(c) = 0$, then by the Second Derivative Test, it follows that f achieves neither a local maximum nor a local minimum at $x = c$.

T **F** The absolute maximum value of a continuous function $f(x)$ defined on a closed interval $[a, b]$ can only be realized at an endpoint ($x = a$ or $x = b$) or at a point where the graph of f has a horizontal tangent.

T **F** If h is continuous, decreasing, and concave down for all x, then $h(x)$ must be negative for some sufficiently large value of x.

T **F** A recommended initial "guess" when using Newton's method to solve the equation $f(x) = 0$ is an x_1 such that $f'(x_1)$ is very close to zero.

T **F** If g is an even function that is continuous at all values, then $\int_{-a}^{a} xg(x)\,dx = 0$ for any value of a.

Formulas for Reference

Geometric Formulas:

$$V_{\text{sphere}} = \frac{4}{3}\pi r^3, \qquad SA_{\text{sphere}} = 4\pi r^2, \qquad A_{\text{circle}} = \pi r^2$$

$$V_{\text{cylinder}} = \pi r^2 h, \qquad V_{\text{cone}} = \frac{1}{3}\pi r^2 h$$

Summation Formulas:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$