Math 41 — Hints for calculating limits

This is a summary of things you should be thinking about when faced with a limit calculation of the form $\lim_{x\to a} f(x)$. You can use a modified version of this checklist (omitting some of the irrelevant items) when faced with a limit of the form $\lim_{x\to\infty} f(x)$.

General advice: we don't want you to have to memorize this list in order to successfully do problems on your own! Instead, practice with text problems and take note of the general pattern of issues that arise; compare with this checklist to see if you've overlooked any other issues. Note that there can often be many ways to solve a particular problem, and this checklist can't capture all those possibilities.

1. ("Plugging in") If f is *continuous* at a, then by definition $\lim_{x\to a} f(x) = f(a)$ (often known as the substitution rule); keep in mind the long list of functions (see pp.115-119) that are known to be continuous at every point in their domain, plus the long list of ways that we can build continuous functions out of other continuous functions (addition, multiplication, composition, etc.).

2. (Indeterminate forms) What if analyzing the expression $\lim_{x\to a} f(x)$ or $\lim_{x\to\infty} f(x)$ appears to lead to an expression of the form
$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty - \infty, \quad \text{or} \quad 0\cdot\infty?$$
These four are *indeterminate forms* and are not only meaningless as arithmetic expressions, but you *cannot* use them to conclude that the limit does not exist! Instead, you must do further analysis (often algebraic manipulation, see below) before you can conclude an answer. (Any manipulation of one expression "$f(x)$" into another "$g(x)$" is legal, provided $f(x) = g(x)$ whenever $x \neq a$, since in this case $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$.)

   Examples of algebraic manipulations:

   - factoring, LCDs: see 2.3 Example 5, p.108; 2.3 #9; 2.5 Example 9, p.132
   - rationalizing radicals: see 2.3 Example 6, p.108; 2.3 #21; 2.5 Example 6, p.131; 2.5 #27
   - multiplying by "1": see 2.5 Example 5, p.130; 2.5 #23

   Later, in section 4.5, you will use l'Hôpital's Rule to extend your ability to handle these four indeterminate forms (plus three others).

3. Remember that $\lim_{x\to a} f(x)$ exists if and only if both the one-sided limits
$$\lim_{x\to a^+} f(x) \text{ and } \lim_{x\to a^-} f(x) \text{ exist and are equal to each other.}$$
In particular, if f is defined piecewise, and the formulas for f are different for $x \to a^+$ and $x \to a^-$, then the one-sided limits must be checked separately, even if each formula is itself a continuous function. [see 2.3 Example 8, p.109, 2.4 #35]

4. Does $\lim_{x \to a} f(x)$ involve infinity?

- A helpful two-step rule for quotients $f(x) = \dfrac{g(x)}{h(x)}$, not provided explicitly by Stewart:
  (a) ("Is infinity involved?") If $\lim_{x \to a} g(x) = L \neq 0$ and $\lim_{x \to a} h(x) = 0$, then you can be sure that the *absolute value* of the quotient grows to infinite size: $\lim_{x \to a} \left| \dfrac{g(x)}{h(x)} \right| = \infty$.
  (b) ("Which: $+\infty$ or $-\infty$?") You must analyze the *sign* of $\frac{g}{h}$ separately, by analyzing the signs of g and h. [see 2.5 Example 1, p.126; 2.5 #17,21]
- May relate to an asymptote (vertical or horizontal) of a "building-block" function or its inverse.
  (a) Any quantity of the form $\ln t$ for $t \to 0$ (so long as $t > 0$) approaches $-\infty$. [see 2.5 #19] Remember that since $\ln t$ is the inverse funtion to e^t, the vertical asymptote in the graph of $\ln t$ is explained by the horizontal asymptote in the graph of e^t; see page 68.
  (b) For analogous reasons, the function $\arctan x = \tan^{-1} x$, which is the inverse function to $\tan x$ $(-\pi/2 < x < \pi/2)$, has two horizontal asymptotes; compare graphs on pp. 34, 219.
  (c) Go back to the list of "basic functions to know" on the Math 41 precalculus checklist from week 1. For each function listed, and its inverse function (if applicable), know the locations of any asymptotes, and which shapes exhibit growth to infinity as $x \to \pm\infty$.

5. Can the Squeeze Theorem be applied? [see 2.3 Example 10, p.110; 2.3 #29; 2.5 #33 (for limits at infinity)]

6. Can the behavior of $f(x)$ near $x = a$ (or, if relevant, for large values of x) be used to show that the limit does not exist? [see 2.2 Example 4, p.98; 2.5 Example 8, p.131]

7. Limit laws for infinite limits — not provided explicitly by Stewart. You should learn these rules *without* memorizing them; instead, learn to think about *why* they are true. (If you replace each "$x \to a$" with "$x \to \infty$" you get another set of valid rules.)

- If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = L$, then $\lim_{x \to a} (f(x) + g(x)) = \infty$. (Here, we mean L is *finite*.)
- If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} (f(x) + g(x)) = \infty$.
  [Why is there no corresponding rule for subtraction?]
- If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = L > 0$, then $\lim_{x \to a} f(x)g(x) = \infty$.
  [There is a corresponding rule for $L < 0$, with $-\infty$ as the result. Why no rule at all for $L = 0$?]
- If $\lim_{x \to a} f(x) = \infty$ and $\lim_{x \to a} g(x) = \infty$, then $\lim_{x \to a} f(x)g(x) = \infty$.
  [Why is there no corresponding rule for division?]
- If $\lim_{x \to a} f(x) = \infty$ (or $-\infty$), then $\lim_{x \to a} \dfrac{1}{f(x)} = 0$.
- If $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = 0$, then the *absolute value* of the quotient $\frac{f(x)}{g(x)}$ will grow to infinity, but the *sign* of the quotient must be analyzed more carefully, by assessing the signs of f and g separately.