Example of domains and inverses

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September 29, 2011

Here is an example of a "domain and inverse" problem to resolve the confusion over Problem 55 in §1.6.

Let

$$f(x) = \sqrt{10 - e^x}.$$

Find (a) the domain of f, and (b) f^{-1} (including its domain!). What is the range of f?

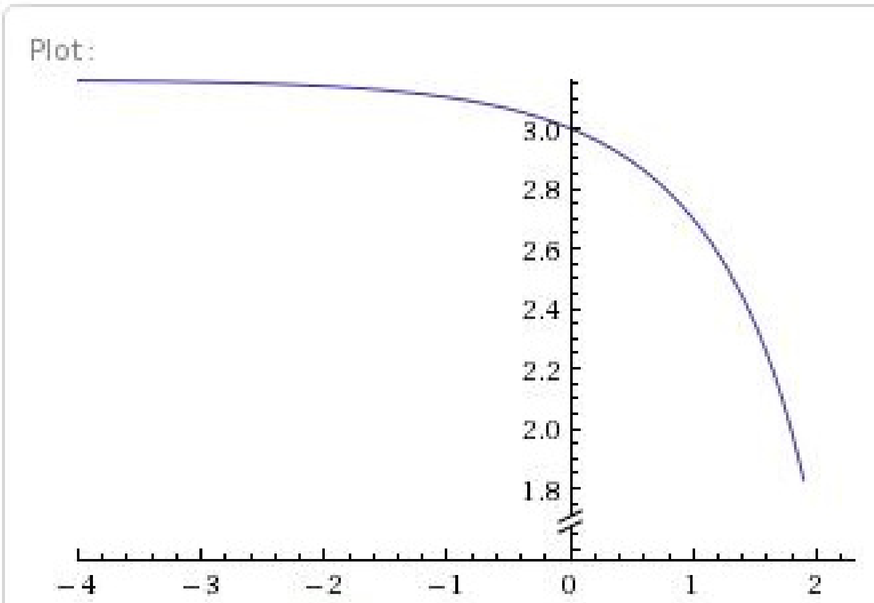
Solution.

(a) To find the domain of f we must determine where the value inside the square root is non-negative.

$$10 - e^x \geq 0 \Leftrightarrow e^x \leq 10 \Leftrightarrow x \leq \ln 10.$$

So the domain of f is $(-\infty, \ln(10)]$.

For clarity, below is a plot of f:



(b) [First, f is one-to-one, so f^{-1} makes sense, because for any x_1, x_2 in the domain of f, if

$$f(x_1) = f(x_2)$$

then

$$\sqrt{10 - e^{x_1}} = \sqrt{10 - e^{x_2}}.$$

So, squaring both sides,

$$10 - e^{x_1} = 10 - e^{x_2}.$$

Simplifying,

$$e^{x_1} = e^{x_2}.$$

This forces $x_1 = x_2$, because e^x is a 1-1 function (since it has an inverse, the natural logarithm). Thus $f(x_1) = f(x_2)$ forces $x_1 = x_2$, which means $f(x)$ is 1-1.]

To find f^{-1} we solve for x in the expression

$$y = \sqrt{10 - e^x}$$

$$y^2 = 10 - e^x$$

$$e^x = 10 - y^2$$
$$x = \ln(10 - y^2).$$

Thus we would *like* to say $f^{-1}(x) = \ln(10-x^2)$, where the righthand side is defined. However, in fact to determine the domain of f^{-1} we must find the range of f. (That is, $f^{-1}(y)$ makes sense if and only if $y = f(x)$ for some x in the domain of f.)

To do this, note that the domain of f is $(-\infty, \ln(10]$. As x ranges from $-\infty$ to $\ln(10)$, we can see from the graph of e^x that e^x ranges from 0 to $e^{\ln(10)} = 10$. (Since e^x is always positive, $e^x > 0$ even for x very far below zero, like $x = -10,000,000$. So as x goes to $-\infty$, e^x approaches 0 but never equals zero.) So for x in the domain of f, $10 - e^x$ ranges from $10 - 0$ to $10 - 10$, i.e. from 10 down to 0. Thus $\sqrt{10 - e^x}$ ranges from $\sqrt{10}$ down to zero, as shown in the graph.

To summarize, $\text{range}(f) = \text{domain}(f^{-1}) = [0, \sqrt{10})$.

So $f^{-1}(x) = \ln(10 - x^2), 0 \leq x < \sqrt{10}$.

Note that the *formula* $\ln(10-x^2)$ defines a function $g(x)$ with domain $-\sqrt{10} < x < \sqrt{10}$. To see this, solve for the values of x where $10 - x^2 > 0$. So the righthand side of the equation

$$f^{-1}(x) = \ln(10 - x^2)$$

is defined for all $x \in (-\sqrt{10}, \sqrt{10})$. But the lefthand side is not! The *function* f^{-1} is not defined for $-\sqrt{10} < x < 0$, since these negative numbers are not in the range of f.

Moral: Keep track of the difference between a ***function*** *and the* ***formula*** *used to describe the function!*

Remark. For a similar example, see Example 5 in §1.6 of the book. In that case $f(x) = \sqrt{-1-x}, x \leq -1$. The inverse function is$f^{-1}(x) = -x^2 - 1, x \geq 0$. Note that the *formula* $x^2 - 1$ makes sense for any value of x. But the *function* f^{-1} is not defined for negative values of x, since these are not in the range of f (because in the definition of f, we use the positive square root).