Solutions to Problems in Handout 20

1. Let $f(x) = x^2 - 3x + 2$, $g(x) = x^3 + x^2 + 1$.
(a) Show that two graphs $y = f(x)$ and $y = g(x)$ intersect at some point with x-coordinate between 0 and 1.

$f(x) = g(x)$ if there is an intersection, so let $h(x) = g(x) - f(x) = x^3 + 3x - 1$.

Then h is continuous, $h(0) = -1$, $h(1) = 3$.

Therefore, by Intermediate Value Theorem, there is c such that $h(c) = 0$ and $0 < c < 1$. Thus $(c, f(c)) = (c, g(c))$ is the point of intersection and $0 < c < 1$ by our choice. ∎

(b) Using Newton's Method with initial value $x_1 = 0$, find a sequence that approaches to x-coordinate of an intersection point.

The x-value of an intersection point is a zero of h. Newton's Method is to find the sequence by the recursion that x_{n+1} is the x-intercept of the tangent line at $x = x_n$. The tangent line to the graph $y = h(x)$ at $x = x_n$ is $y = h'(x_n)(x - x_n) + h(x_n)$, so the x-intercept is $x = x_n - \frac{h(x_n)}{h'(x_n)}$. Therefore the sequence is

$$x_{n+1} = x_n - \frac{x_n^3 + 3x_n - 1}{3x_n^2 + 3}, \qquad x_1 = 0.$$

∎

2. The value 1 is a pretty lousy approximation to $\sqrt[3]{2}$. Using Newton's method, find a rational number (whole number or fraction) which is closer, and then another one which is still closer.

$\sqrt[3]{2}$ is the zero of the equation $x^3 - 2 = 0$. Applying Newton's Method on $f(x) = x^3 - 2$, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 2}{3x_n^2} = \frac{2}{3}x_n + \frac{2}{3x_n^2}.$$

Then if you start with the lousy approximation $x_1 = 1$, we have $x_2 = \frac{2}{3} + \frac{2}{3} = \frac{4}{3}$, which is a better approximation.

$$x_3 = \frac{2}{3}x_2 + \frac{2}{3x_2^2} = \frac{2}{3} \cdot \frac{4}{3} + \frac{2}{3} \cdot \frac{9}{16} = \frac{91}{72}$$

is even better approximation. ∎

3. Find
$$\lim_{x\to 3} \frac{x}{x-3}\int_3^x \frac{\sin t}{t}dt$$

We have $\int_3^x \frac{\sin t}{t}dt \to 0$ as $x \to 3$. Then the function inside the limit is $x \cdot \frac{1}{x-3}\int_3^x \frac{\sin t}{t}dt$ and the limit of the second factor becomes

$$\lim_{x\to 3}\frac{\int_3^x \frac{\sin t}{t}dt}{x-3} = \lim_{x\to 3}\frac{(\int_3^x \frac{\sin t}{t}dt)'}{(x-3)'}$$

by L'hopital theorem. Since $\frac{d}{dx}\int_3^x \frac{\sin t}{t}dt = \frac{\sin x}{x}$ by Fundamental Theorem of Calculus, we have

$$\lim_{x\to 3}\frac{\int_3^x \frac{\sin t}{t}dt}{x-3} = \lim_{x\to 3}\frac{\frac{\sin x}{x}}{1} = \frac{\sin 3}{3}$$

Therefore the limit is

$$\begin{aligned}\lim_{x\to 3}\frac{x}{x-3}\int_3^x \frac{\sin t}{t}dt &= \lim_{x\to 3} x\cdot\frac{\int_3^x \frac{\sin t}{t}dt}{x-3} \\ &= \lim_{x\to 3} x \lim_{x\to 3}\frac{\int_3^x \frac{\sin t}{t}dt}{x-3} = 3\cdot\frac{\sin 3}{3} = \sin 3\end{aligned}$$

■

4. Suppose all that is known about the function f is that $-x^2+x+1 \leq f(x) \leq x^2+x+1$ for all x.
(a) Find $f(0)$.

Substituting $x=0$, we have $1 \leq f(0) \leq 1$. Therefore $f(0)=1$ ■

(b) Determine whether f is differentiable at $x=0$, using the limit definition of the derivative.

f is differentiable at $x=0$ if and only if $\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$ exists. $f(0)=1$ from part (a), so let $h(x)=(f(x)-1)/x$. From the inequality, $-x^2+x \leq f(x)-1 \leq x^2+x$ and when $x>0$,

$$-x+1 \leq \frac{f(x)-1}{x} = h(x) \leq x+1.$$

If we take the limit as $x \to 0^+$ of the right and the leftmost functions, $\lim_{x\to 0^+}(-x+1) = \lim_{x\to 0^+}(x+1) = 1$. By Squeeze Theorem on this inequality, we have $\lim_{x\to 0^+} h(x) = 1$. On the other hand, when $x<0$, we have $-x+1 \geq h(x) \geq x+1$ and by Squeeze Theorem again, we have $\lim_{x\to 0^-} h(x) = 1$. Therefore $\lim_{x\to 0} h(x) = 1$ and f is differentiable at $x=0$. ■

(c) Show that if f is integrable, $\frac{3}{2} \le \int_{-1}^{2} f(x) \le \frac{15}{2}$.

$$\int_{-1}^{2} -x^2 + x + 1 dx \le \int_{-1}^{2} f(x) dx \le \int_{-1}^{2} x^2 + x + 1 dx$$

and $\int_{-1}^{2} -x^2 + x + 1dx = \left[\frac{1}{3}x^3 + \frac{1}{2}x^2 + x\right]_{-1}^{2} = \frac{1}{3}(2^3 - (-1)^3) + \frac{1}{2}(4-1) + (2-(-1)) = \frac{3}{2}$, $\int_{-1}^{2} x^2 + x + 1dx = 3 + \frac{3}{2} + 3 = \frac{15}{2}$. So the inequality follows.■

5. On the flat ground, sand was piling at a rate $\pi(27 - t^2)$ (cm^3/s) at time $0 \le t \le 5$. The pile forms the shape of a cone in a way that the base radius equals the height. How fast does the height of the pile increase at $t = 3$?

Let V be the volume of the pile, h be the height of the pile and $r = h$ be the base radius. Then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi h^3$ as the volume of a cone.

The rate of change $\frac{dV}{dt} = \pi(27 - t^2)$, and we want to find $\frac{dh}{dt}$ at $t = 3$. Differentiating the relation $V = \frac{1}{3}\pi h^3$,

$$\frac{dV}{dt} = \frac{1}{3}\pi \cdot 3h^2 \frac{dh}{dt} = \pi h^2 \frac{dh}{dt}.$$

$\frac{dV}{dt}|_{t=3} = \pi(27 - 3^2) = 18\pi$, so we need to find h at $t = 3$.

Since V is an antiderivative of $\pi(27 - t^2)$, $V = \pi(27t - \frac{1}{3}t^3) + C$ for some constant C. At $t = 0$, the pile was empty, so we have $C = 0$. Then at $t = 3$, $V = \pi(81 - 9) = 72\pi = \frac{1}{3}\pi h^3$ and thus $h = 6$. Therefore $18\pi = \frac{dV}{dt}|_{t=3} = \pi \cdot 6^2 \frac{dh}{dt}|_{t=3}$. Therefore $\frac{dh}{dt} = 0.5(cm/s)$ ■

6. Determine $\frac{d}{dx} \int_{\arctan x}^{x^2} 2^t dt$.

Let $F(t)$ be the antiderivative of 2^t. Then the expression is $\frac{d}{dx}(F(x^2) - F(\arctan x)) = F'(x^2) \cdot (2x) - F'(\arctan x) \cdot \frac{1}{1+x^2}$ by Chain rule. Using $F'(t) = 2^t$ the expression is

$$2^{x^2} \cdot 2x - 2^{\arctan x} \cdot \frac{1}{1+x^2}$$

■

7. Find f and a such that $\int_a^x f(t)dt = 4\sqrt{9 + x^2} - 16$.

Differentiating both sides, we have $(\int_a^x f(t)dt)' = f(x)$ by Fundamental Theorem of Calculus and $(4\sqrt{x^2+9} - 16)' = 4x/\sqrt{x^2+9}$ Therefore $f(x) = 4x/\sqrt{x^2+9}$. To find a, set $x = a$ and we have

$$0 = \int_a^a f(x)dx = 4\sqrt{9 + a^2} - 16.$$

Solving this equation, we have $a = \pm\sqrt{7}$. ■

8. Suppose f is a continuous function and $\int_0^3 f(x)dx = 5$.
(a) Show that $\int_0^{c_1} f(x)dx = 2$ for some $0 < c_1 < 3$.

Let $h(x) = \int_0^x f(t)dt$. Then h is continuous, $h(0) = \int_0^0 f(x)dx = 0$, $h(3) = 5$ by assumption. Therefore, by Intermediate Value Theorem, there is c_1 with $0 < c_1 < 3$ such that $h(c_1) = 2$. Thus $h(c_1) = \int_0^{c_1} f(x)dx = 2$. ∎

(b) Show that $f(c_2) = \frac{5}{3}$ for some $0 < c_2 < 3$

h is differentiable with derivative $h'(x) = f(x)$ by Fundamental Theorem of Calculus. Using Mean Value Theorem, there is c_2 with $0 < c_2 < 3$ such that

$$h'(c_2) = \frac{h(3) - h(0)}{3 - 0} = \frac{5}{3}.$$

Thus $f(c_2) = h'(c_2) = \frac{5}{3}$ as desired. ∎