

GRAPHING PRACTICE

(1) For the function $f(x) = \frac{x^2 - 1}{x^3}$

(a) Find the domain.

$$(-\infty, 0) \cup (0, \infty)$$

(b) Where is $f(x)$ increasing? Decreasing? Find all local max/min.

$$f'(x) = \frac{2x \cdot x^3 - 3x^2(x^2 - 1)}{x^6} = \frac{2x^4 - 3x^4 + 3x^2}{x^6} = \frac{-x^4 + 3x^2}{x^6} = \frac{-x^2 + 3}{x^4}$$

$$f'(x) = 0 \Rightarrow -x^2 + 3 = 0 \Rightarrow x = \sqrt{3}, -\sqrt{3}.$$

We also need to consider $x = 0$ because the function is not continuous there.

$$f'(-2) = \frac{-4 + 3}{(-2)^4} < 0$$

$$f'(-1) = \frac{-1 + 3}{1} > 0$$

$$f'(1) = \frac{-1 + 3}{1} > 0$$

$$f'(2) = \frac{-4 + 3}{2^4} < 0$$

So we have a table:

f	Decreasing	Increasing	Increasing	Decreasing
	<div style="display: flex; justify-content: space-around; width: 100%;"> </div>			
	$-\sqrt{3}$	0	$\sqrt{3}$	
f'	-	+	+	-

$f(x)$ is increasing on $(-\sqrt{3}, 0) \cup (0, \sqrt{3})$

$f(x)$ is decreasing on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

There is a local minimum at the point $(-\sqrt{3}, f(-\sqrt{3})) = (-\sqrt{3}, \frac{2}{-3\sqrt{3}})$

There is a local maximum at the point $(\sqrt{3}, f(\sqrt{3})) = (\sqrt{3}, \frac{2}{3\sqrt{3}})$

(c) Where is $f(x)$ concave up? Concave down? Find all inflection points.

$$f''(x) = \frac{-2x \cdot x^4 - 4x^3(-x^2 + 3)}{x^8} = \frac{-2x^2 - 4(-x^2 + 3)}{x^5} = \frac{-2x^2 + 4x^2 - 12}{x^5} = \frac{2x^2 - 12}{x^5}$$

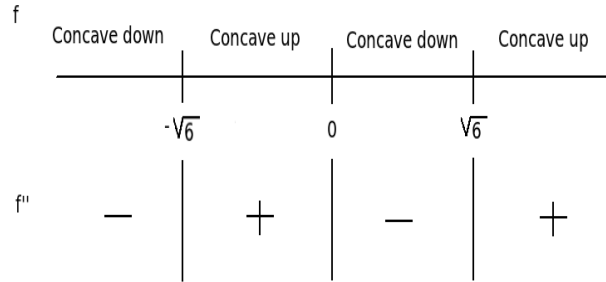
$$f''(x) = 0 \Rightarrow 2x^2 - 12 = 0 \Rightarrow x = \sqrt{6}, -\sqrt{6}$$

$$f''(-3) = \frac{18 - 12}{(-3)^5} < 0$$

$$f''(-1) = \frac{2 - 12}{-1} > 0$$

$$f''(1) = \frac{2 - 12}{1} < 0$$

$$f''(3) = \frac{18 - 12}{3^5} > 0$$



$f(x)$ is concave down on $(-\infty, -\sqrt{6}) \cup (0, \sqrt{6})$

$f(x)$ is concave up on $(-\sqrt{6}, 0) \cup (\sqrt{6}, \infty)$

There is two inflection points, $(-\sqrt{6}, f(-\sqrt{6})) = (-\sqrt{6}, \frac{5}{-6\sqrt{6}})$ and $(\sqrt{6}, f(\sqrt{6})) = (\sqrt{6}, \frac{5}{6\sqrt{6}})$.

(d) Find all the asymptotes of $f(x)$

-Horizontal

$$\lim_{x \rightarrow \infty} \frac{x^2 - 1}{x^3} = \lim_{x \rightarrow \infty} \frac{1/x - 1/x^3}{1} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x^3} = \lim_{x \rightarrow -\infty} \frac{1/x - 1/x^3}{1} = 0$$

There is a horizontal asymptote, $y = 0$

-Vertical

The only candidate for a vertical asymptote is $x = 0$ because $f(x)$ is continuous on $(-\infty, 0) \cup (0, \infty)$, being a rational function whose denominator only vanishes at $x = 0$.

$$\lim_{x \rightarrow 0^-} \frac{x^2 - 1}{x^3} = \infty$$

because the top goes to -1 and the bottom to 0 . The sign is positive because when $x < 0$ near $x = 0$, $x^3 < 0$ and $x^2 - 1$ is near -1 and so $x^2 - 1 < 0$. and the denominator is only zero for $x = 0$.

$$\lim_{x \rightarrow 0^-} \frac{x^2 - 1}{x^3} = \infty$$

because the top goes to -1 and the bottom to 0 . The sign is positive because when $x < 0$ near $x = 0$, $x^3 < 0$ and $x^2 - 1$ is near -1 and so $x^2 - 1 < 0$.

So $x = 0$ is a vertical asymptote. Let us do the other side limit because it will help for the graphing.

$$\lim_{x \rightarrow 0^+} \frac{x^2 - 1}{x^3} = -\infty$$

The sign is negative because when $x > 0$ near $x = 0$, $x^3 > 0$ and $x^2 - 1$ is near -1 and so $x^2 - 1 < 0$.

- (e) Find the intercepts of $f(x)$, if any.

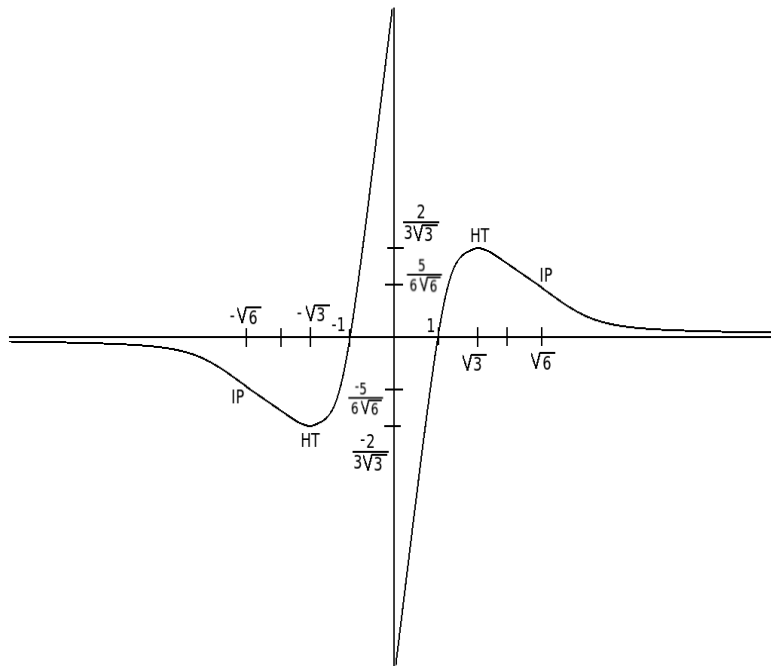
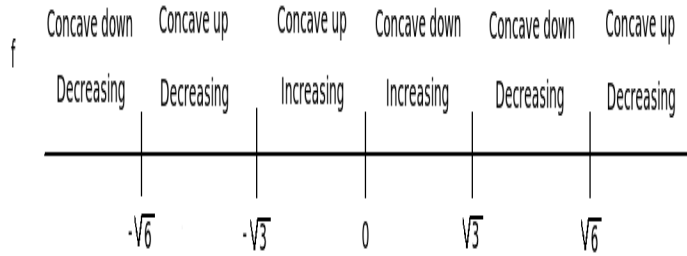
$f(x) = 0$ when $x^2 - 1 = 0 \Rightarrow x = 1, -1$. So the x -intercepts are $(1, 0)$ and $(-1, 0)$.

There is no y -intercept because $x = 0$ is not on the domain.

- (f) Where is $f(x)$ continuous? Where differentiable?

Rational functions are continuous and differentiable on their domains, so $f(x)$ is differentiable on $(-\infty, 0) \cup (0, \infty)$.

- (g) Sketch the graph of $f(x)$



(2) For the function $f(x) = \frac{\ln(x)}{x}$

(a) Find the domain.

$(0, \infty)$

(b) Where is $f(x)$ increasing? Decreasing? Find all local max/min.

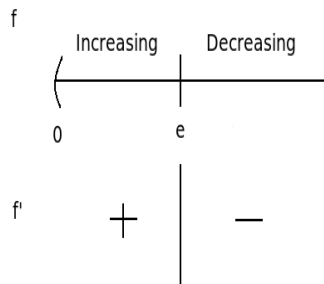
$$f'(x) = \frac{1/x \cdot x - \ln(x)}{x^2} = \frac{1 - \ln(x)}{x^2}$$

$$f'(x) = 0 \Rightarrow 1 - \ln(x) = 0 \Rightarrow x = e.$$

The derivative is defined elsewhere in the domain.

$$f'(1) = \frac{1}{1} > 0$$

$$f'(e^2) = \frac{1 - 2}{e^4} < 0$$



$f(x)$ is increasing on $(0, e)$

$f(x)$ is decreasing on (e, ∞)

There is a local maximum at the point $(e, f(e)) = \left(e, \frac{1}{e}\right)$

(c) Find all the asymptotes of $f(x)$. You can use $\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = 0$

-Horizontal

$$\lim_{x \rightarrow \infty} \frac{\ln(x)}{x} = 0$$

So $y = 0$ is a horizontal asymptote.

It does not make sense to do $\lim_{x \rightarrow -\infty} f(x)$ because the domain is $(0, \infty)$

-Vertical

The only candidate for a vertical asymptote is $x = 0$ because $f(x)$ is continuous on $(0, \infty)$ being the quotient of a logarithmic function and a polynomial for which the denominator is not zero.

$$\lim_{x \rightarrow 0^+} \frac{\ln(x)}{x} = -\infty$$

because the top goes to $-\infty$ and the bottom goes to 0 and it is positive.

So $x = 0$ is a vertical asymptote. It does not make sense to do $\lim_{x \rightarrow 0^-} f(x)$ because the domain is $(0, \infty)$

(d) Find the intercepts of $f(x)$, if any.

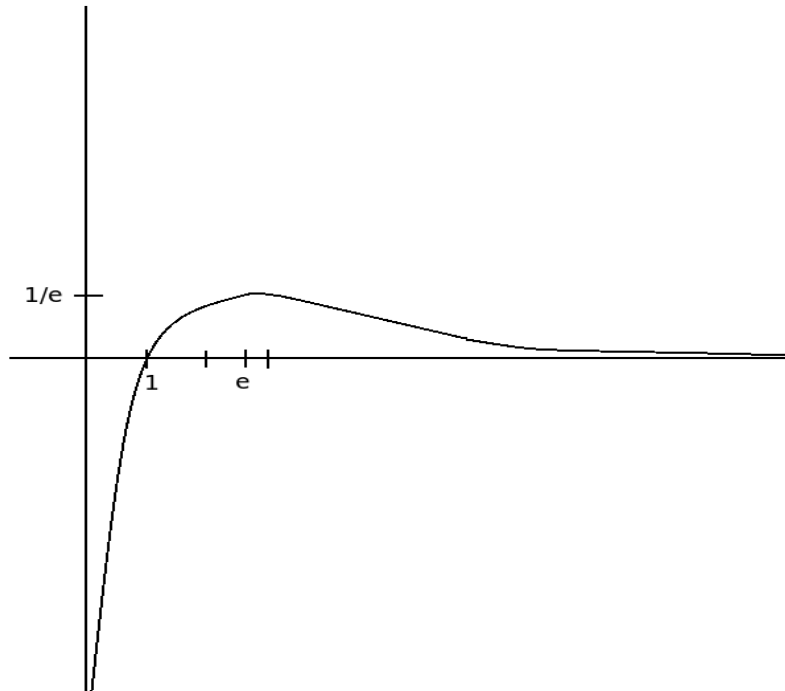
$f(x) = 0$ when $\ln(x) = 0 \Rightarrow x = 1$. So the only x -intercept is $(1, 0)$.

There is no y -intercept because $x = 0$ is not on the domain.

(e) Is $f(x)$ differentiable on $(0, \infty)$?

Yes, because it is the quotient of a logarithmic function and a polynomial, which are differentiable, for which the denominator is not zero.

(f) Sketch the graph of $f(x)$



(3) For the function $f(x) = \frac{|x| - 2}{x^2 - 4}$

(a) Find the domain.

$$(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$$

(b) Where is $f(x)$ increasing? Decreasing? Find all local max/min.

$$f(x) = \begin{cases} \frac{-x-2}{x^2-4} = \frac{(-1)(x+2)}{(x-2)(x+2)} = \frac{-1}{x-2} & \text{if } x \leq 0, x \neq -2 \\ \frac{x-2}{x^2-4} = \frac{1}{x+2} & \text{if } x \geq 0, x \neq 2 \end{cases}$$

$$\text{If } x \neq 0, 2, -2, \text{ then } f'(x) = \begin{cases} \frac{1}{(x-2)^2} & \text{if } x < 0 \\ \frac{-1}{(x+2)^2} & \text{if } x > 0 \end{cases}$$

At $x = 0$, $f(x)$ may not be differentiable, we will check that later with the definition of the derivative. Since it may give problems, it is better to include it for now in our diagrams.

We notice that $f'(x) > 0$ if $x < 0$ and $f'(x) < 0$ if $x > 0$.

f	Increasing	Increasing	Decreasing	Decreasing
	-2	0	2	
f'	+	+	-	-

$f(x)$ is increasing on $(0, 2) \cup (2, \infty)$

$f(x)$ is decreasing on $(-\infty, -2) \cup (-2, 0)$

There is a local maximum at the point $(0, f(0)) = \left(0, \frac{-1}{2}\right)$

(c) Where is $f(x)$ concave up? Concave down? Find all inflection points.

$$\text{If } x \neq 0, 2, -2, \text{ then } f''(x) = \begin{cases} \frac{-2}{(x-2)^3} & \text{if } x < 0 \\ \frac{2}{(x+2)^3} & \text{if } x > 0 \end{cases}$$

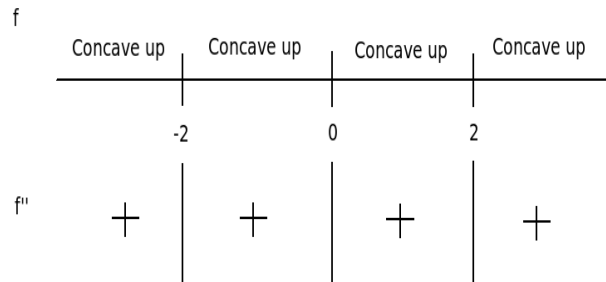
Again, $f''(x)$ can not be zero, except maybe at $x = 0$.

$$f''(-3) = \frac{-2}{(-5)^3} > 0$$

$$f''(-1) = \frac{-2}{-1} > 0$$

$$f''(1) = \frac{2}{3^3} > 0$$

$$f''(3) = \frac{2}{5^3} > 0$$



$f(x)$ is concave up on $(-\infty, -2) \cup (-2, 0) \cup (0, 2) \cup (2, \infty)$

There are no inflection points.

(d) Find all the asymptotes of $f(x)$

-Horizontal

$$\lim_{x \rightarrow \infty} f(x) = \lim_{x \rightarrow \infty} \frac{1}{x+2} = 0$$

$$\lim_{x \rightarrow -\infty} f(x) = \lim_{x \rightarrow -\infty} \frac{-1}{x-2} = 0$$

There is a horizontal asymptote, $y = 0$

-Vertical

The candidates for a vertical asymptote are $x = 2, -2$ because $f(x)$ is continuous elsewhere, being a quotient of combinations of absolute values and polynomials, which are continuous as long as the denominator is not zero.

$$\lim_{x \rightarrow 2} \frac{|x| - 2}{x^2 - 4} = \lim_{x \rightarrow 2} \frac{1}{x+2} = \frac{1}{4}$$

the first equality because near $x = 2$, $f(x) = \frac{1}{x+2}$.

$$\lim_{x \rightarrow -2} \frac{|x| - 2}{x^2 - 4} = \lim_{x \rightarrow -2} \frac{-1}{x - 2} = \frac{1}{4}$$

the first equality because near $x = -2$, $f(x) = \frac{-1}{x-2}$.

So there are no vertical asymptotes

(e) Find the intercepts of $f(x)$, if any.

$$f(0) = \frac{1}{2}$$

So the y -intercept is $\left(0, \frac{1}{2}\right)$.

$f(x) = 0 \Rightarrow |x| - 2 = 0 \Rightarrow x = 2, -2$. Not on the domain, so no x -intercepts.

(f) Where is $f(x)$ differentiable?

$f(x)$ is differentiable apart from 2, -2 and 0 because it is a quotient of combinations of absolute values and polynomials, which are differentiable as long as the denominator is not zero and the inside of the absolute value is not zero.

$f(x)$ is not differentiable at 2 or -2 because it is not continuous there.

Let us study the differentiability at $x = 0$.

$$\begin{aligned} \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \rightarrow 0^+} \frac{1/(x+2) - 1/2}{x} = \lim_{x \rightarrow 0^+} \frac{2 - (x+2)}{2x(x+2)} = \lim_{x \rightarrow 0^+} \frac{-x}{2x(x+2)} = \\ &= \lim_{x \rightarrow 0^+} \frac{-1}{2(x+2)} = \frac{-1}{4} \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} &= \lim_{x \rightarrow 0^-} \frac{-1/(x-2) - 1/2}{x} = \lim_{x \rightarrow 0^-} \frac{-2 - (x-2)}{2x(x-2)} = \lim_{x \rightarrow 0^-} \frac{-x}{2x(x-2)} = \\ &= \lim_{x \rightarrow 0^-} \frac{-1}{2(x-2)} = \frac{1}{4} \end{aligned}$$

So $f(x)$ is not differentiable at $x = 0$, there is a corner there.

(g) Sketch the graph of $f(x)$

