Math 220B - Summer 2003
Homework 7 Solutions

1. Consider the eigenvalue problem,

$$\begin{cases} -\Delta w = \lambda w & x \in \Omega \\ \frac{\partial w}{\partial n} + a(x)w = 0 & x \in \partial\Omega. \end{cases} \tag{1}$$

Let $\{v_i\}$ be the eigenfunctions for this problem. Let

$$Y_n \equiv \{w \in C^2 : w \not\equiv 0, \langle w, v_i\rangle = 0 \text{ for } i = 1, \ldots, n-1\}.$$

Let

$$J(w) \equiv \left\{ \frac{\int_\Omega |\nabla w|^2\,dx + \int_{\partial\Omega} a(x)w^2\,dS(x)}{\int_\Omega w^2\,dx} \right\}.$$

Suppose there exists a function $u_n \in Y_n$ such that

$$J(u_n) = \min_{w \in Y_n} J(w).$$

Let $m_n \equiv J(u_n)$. Show that m_n is the n^{th} eigenvalue of (1) with corresponding eigenfunction u_n.

Answer: By assumption, $u_n \in Y_n$ is the minimizer of $J(w)$ over all $w \in Y_n$. Therefore,

$$m_n \equiv J(u_n) = \frac{\int_\Omega |\nabla u_n|^2\,dx + \int_{\partial\Omega} a(x)u_n^2\,dS(x)}{\int_\Omega u_n^2\,dx} \le \frac{\int_\Omega |\nabla w|^2\,dx + \int_{\partial\Omega} a(x)w^2\,dS(x)}{\int_\Omega w^2\,dx}$$

for all $w \in Y_n$. Let $v \in Y_n$. Note that $u_n + \epsilon v \in Y_n$ for all $\epsilon \in \mathbb{R}$. Let

$$f(\epsilon) \equiv \frac{\int_\Omega |\nabla(u_n + \epsilon v)|^2\,dx + \int_{\partial\Omega} a(x)(u_n + \epsilon v)^2\,dS(x)}{\int_\Omega (u_n + \epsilon v)^2\,dx}.$$

If u_n is the minimizer of J, then $f'(0) = 0$. We calculate $f'(0)$. We have

$$f'(0) = \frac{\left[\int_\Omega u_n^2\right]\left[2\int_\Omega \nabla v \cdot \nabla u_n + 2\int_{\partial\Omega} a(x)u_n v\right] - \left[2\int_\Omega u_n v\right]\left[\int_\Omega |\nabla u_n|^2 + \int_{\partial\Omega} a(x)u_n^2\right]}{\left[\int u_n^2\right]^2}$$

So, $f'(0) = 0 \implies$

$$\begin{aligned} \int_\Omega \nabla u_n \cdot \nabla v + \int_{\partial\Omega} a(x)u_n v &= \frac{\int_\Omega |\nabla u_n|^2 + \int_{\partial\Omega} a(x)u_n^2}{\int_\Omega u^2} \int_\Omega u_n v \\ &= m_n \int_\Omega u_n v. \end{aligned}$$

Integrating by parts, we get

$$-\int_\Omega \Delta u_n v + \int_{\partial\Omega} \frac{\partial u_n}{\partial\nu} v + \int_{\partial\Omega} a(x)u_n v = m_n \int_\Omega u_n v.$$

Therefore,

$$\int_\Omega [\Delta u_n + m_n u_n] v\, dx = \int_{\partial\Omega} \left[\frac{\partial u_n}{\partial \nu} + a(x) u_n\right] v\, dS(x)$$

for all $v \in Y_n$.

Now let v_j be one of the first $n-1$ eigenfunctions for this problem. By assumption, $u_n \in Y_n$ implies that u_n is orthogonal to v_j. Therefore, we see that

$$\begin{aligned}
\int_\Omega [\Delta u + m_n u_n] v_j\, dx &= \int_\Omega \Delta u_n v_j\, dx \\
&= \int_\Omega u_n \Delta v_j\, dx + \int_{\partial\Omega} \frac{\partial u_n}{\partial \nu} v_j\, dS(x) - \int_{\partial\Omega} u_n \frac{\partial v_j}{\partial \nu}\, dS(x) \\
&= -\lambda_j \int_\Omega u_n v_j\, dx + \int_{\partial\Omega} \frac{\partial u_n}{\partial \nu} v_j\, dS(x) + \int_{\partial\Omega} u_n a(x) v_j\, dS(x) \\
&= \int_{\partial\Omega} \left[\frac{\partial u_n}{\partial \nu} + a(x) u_n\right] v_j\, dS(x).
\end{aligned}$$

Now let h be an arbitrary C^2 function on Ω Define

$$w \equiv h - \sum_{i=1}^{n-1} c_i v_i$$

where

$$c_i \equiv \frac{\langle h, v_i\rangle}{\langle v_i, v_i\rangle}.$$

First, we note that

$$\langle w, v_j\rangle = \langle h - \sum_{i=1}^{n-1} c_i v_i\rangle = 0.$$

Therefore, we conclude that $w \in Y_n$.

Therefore,

$$\begin{aligned}
\int_\Omega [\Delta u_n + m_n u_n] h\, dx &= \int_\Omega [\Delta u_n + m_n u_n] \left\{ w + \sum_{i=1}^{n-1} c_i v_i \right\} dx \\
&= \int_\Omega [\Delta u_n + m_n u_n] w\, dx + \sum_{i=1}^{n-1} c_i \int_\Omega v_i\, dx \\
&= \int_{\partial\Omega} \left[\frac{\partial u_n}{\partial \nu} + a(x) u_n\right] w\, dS(x) \\
&\qquad + \sum_{i=1}^{n-1} c_i \int_{\partial\Omega} \left[\frac{\partial u_n}{\partial \nu} + a(x) u_n\right] v_i\, dS(x) \\
&= \int_{\partial\Omega} \left[\frac{\partial u_n}{\partial \nu} + a(x) u_n\right] h\, dS(x).
\end{aligned}$$

To summarize, we have shown that for an arbitrary C^2 function h on Ω,

$$\int_\Omega [\Delta u_n + m_n u_n] h \, dx = \int_{\partial\Omega} \left[\frac{\partial u_n}{\partial \nu} + a(x) u_n \right] h \, dS(x). \tag{2}$$

Now take a C^2 function h which *vanishes* on $\partial\Omega$. For such an h, we have

$$\int_\Omega [\Delta u_n + m_n u_n] h \, dx = 0.$$

Since this is true for all $h \in C^2(\Omega)$ such that $h = 0$ for $x \in \partial\Omega$, we conclude that

$$\Delta u_n + m_n u_n = 0$$

for $x \in \Omega$.

Now we need to check that the boundary condition is satisfied. By (2) and the fact that $\Delta u_n + m_n u_n = 0$, we conclude that

$$\int_{\partial\Omega} \left[\frac{\partial u_n}{\partial \nu} + a(x) u_n \right] h \, dS(x)$$

for all C^2 functions h. Therefore, we conclude that

$$\frac{\partial u_n}{\partial \nu} + a(x) u_n = 0 \qquad x \in \partial\Omega.$$

Therefore, we have shown that u is an eigenfunction with eigenvalue m_n.

Now we need to show that m_n is the n^{th} eigenvalue of our eigenvalue problem. First, we will show that $m_n \geq \lambda_{n-1} \geq \lambda_{n-2} \geq \ldots$. If λ_i is an eigenvalue with eigenfunction v_i for $i \leq n-1$, then

$$\lambda_i \equiv J(v_i) = \min_{w \in Y_i} J(w),$$

But, $Y_n \subset Y_i$ if $i \leq n-1$. Therefore,

$$m_n = J(u_n) \geq J(v_i) = \lambda_i \qquad i \leq n-1.$$

It just remains to show that $m_n \leq \lambda_i$ for $i \geq n+1$. Suppose λ_i is an eigenvalue with eigenfunction v_i for $i \geq n+1$. But $Y_i \subset Y_n$ if $i \geq n+1$. Therefore,

$$m_n = J(u_n) \leq J(v_i) = \lambda_i \qquad i \geq n+1.$$

Therefore, m_n is the n^{th} eigenvalue as claimed.

2. Let $\Omega = \left\{(x,y) \in \mathbb{R}^2 : x^2 + \frac{y^2}{4} < 1\right\}$. Consider the eigenvalue problem with Dirichlet boundary conditions,

$$\begin{cases} -\Delta u = \lambda u & (x,y) \in \Omega \\ u = 0 & (x,y) \in \partial\Omega. \end{cases} \tag{3}$$

Compute the Rayleigh quotient of the trial function $w(x, y) = 4 - 4x^2 - y^2$ to approximate the first eigenvalue of (3). (*Hint: Make the substitution* $x = r\cos(\theta)$, $y = 2r\sin(\theta)$.)

Answer. Making the suggested change of variable, the ellipse is defined in terms of r and θ by $\Omega = \{(r, \theta) \mid 0 \le \theta \le 2\pi, 0 \le r < 1\}$. Also, the Jacobian of this change of variable is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ 2\sin\theta & 2r\cos\theta \end{vmatrix} = 2r$$

so we have $dA = 2r\,dr\,d\theta$. Since $4 - 4x^2 - y^2 = 4 - 4r^2$, it follows that

$$\begin{aligned} \|w\|_{L^2}^2 &= \iint_\Omega (4 - 4x^2 - y^2)^2\,dA \\ &= \int_0^{2\pi}\int_0^1 (4 - 4r^2)^2 2r\,dr\,d\theta \\ &= 2\pi \int_0^1 (4 - 4r^2)^2 2r\,dr \\ &= 2\pi \left[\frac{(4 - 4r^2)^3}{3(-4)}\right]_0^1 \\ &= \frac{32\pi}{3}. \end{aligned}$$

Next, $\nabla w = (-8x, -2y)$, so

$$\|\nabla w\|_{L^2}^2 = \iint_\Omega 64x^2 + 4y^2\,dA = \int_0^{2\pi}\int_0^1 (64r^2\cos^2\theta + 16r^2\sin^2\theta)2r\,dr\,d\theta$$

Exchanging the order of integration and making use of the identities $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ and $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$, this becomes

$$\begin{aligned} \int_0^1\int_0^{2\pi} (64r^2\cos^2\theta + 16r^2\sin^2\theta)2r\,d\theta\,dr &= \pi\int_0^1 (64r^2 + 16r^2)2r\,dr \\ &= 160\pi\int_0^1 r^3\,dr \\ &= 40\pi. \end{aligned}$$

Thus the Rayleigh quotient for w is

$$\frac{\|\nabla w\|_{L^2}^2}{\|w\|_{L^2}^2} = \frac{15}{4}.$$

Since w is a C^2 function and $w = 0$ on $\partial\Omega$, the Minimum Principle for the first eigenvalue implies that $\lambda_1(\Omega) \le \frac{15}{4}$.

3. Consider the eigenvalue problem (3). Let $w_1(x, y) = 4 - 4x^2 - y^2$. Let $w_2(x, y) = (4 - 4x^2 - y^2)^2$. Use the Rayleigh-Ritz approximation method to get an estimate on the first two eigenvalues of (3).

   Answer. By calculations similar to those in the previous problem, we obtain

$$\begin{aligned}
\langle \nabla w_1, \nabla w_1 \rangle &= 40\pi \\
\langle \nabla w_1, \nabla w_2 \rangle &= \frac{320\pi}{3} \\
\langle \nabla w_2, \nabla w_2 \rangle &= \frac{1280\pi}{3} \\
\langle w_1, w_1 \rangle &= \frac{32\pi}{3} \\
\langle w_1, w_2 \rangle &= 32\pi \\
\langle w_2, w_2 \rangle &= \frac{512\pi}{5}.
\end{aligned}$$

   So letting

$$A = \begin{bmatrix} 40\pi & \frac{320\pi}{3} \\ \frac{320\pi}{3} & \frac{1280\pi}{3} \end{bmatrix} \qquad B = \begin{bmatrix} \frac{32\pi}{3} & 32\pi \\ 32\pi & \frac{512\pi}{5} \end{bmatrix}$$

   the Rayleigh-Ritz approximation for $\lambda_1(\Omega)$ and $\lambda_2(\Omega)$ is given by the roots of the quadratic

$$\det(A - \lambda B) = \frac{1024}{15}\lambda^2 - \frac{16384}{9}\lambda + \frac{51200}{9}.$$

   The roots are

$$\lambda_1 = \frac{40 - 5\sqrt{34}}{3} \approx 3.6151 \qquad \lambda_2 = \frac{40 + 5\sqrt{34}}{3} \approx 23.0516.$$

4. Consider the eigenvalue problem (3).

   (a) Find a lower bound on the first eigenvalue of (3) given by a rectangle containing Ω.

   Answer. The rectangle $R = [-1, 1] \times [-2, 2]$ contains Ω. By the Comparison Principle, $\lambda_1(R) \leq \lambda_1(\Omega)$. The eigenvalues of a rectangle with side lengths a and b are

$$\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

   where m and n are positive integers. The first eigenvalue of R is therefore $\lambda_1(R) = \left(\frac{\pi}{2}\right)^2 + \left(\frac{\pi}{4}\right)^2 = \frac{5}{16}\pi^2$, so $\lambda_1(\Omega) \geq \frac{5}{16}\pi^2$.

   (b) Find the best upper bound on the first eigenvalue of (3) given by rectangles inscribed within Ω with sides parallel to the x and y axes.

   Answer. Such a rectangle has one vertex at $(x, 2\sqrt{1 - x^2})$ for some $x \in (0, 1)$. Its dimensions are therefore $a = 2x$ by $b = 4\sqrt{1 - x^2}$. Since the eigenvalues of a rectangle with side lengths a and b are

$$\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

where m and n are positive integers, the first eigenvalue of this rectangle (call it R_x) is

$$\lambda_1(R_x) = \pi^2 \left(\frac{1}{4x^2} + \frac{1}{16(1-x^2)} \right)$$

and by the Comparison Principle $\lambda_1(\Omega) \leq \lambda_1(R_x)$. The best upper bound is obtained by minimizing the above expression over $x \in (0,1)$. Setting $y = x^2$, this expression becomes

$$f(y) = \frac{\pi^2}{16} \left(\frac{4}{y} + \frac{1}{1-y} \right)$$

Since

$$f'(y) = \frac{\pi^2}{16} \left(-\frac{4}{y^2} + \frac{1}{(1-y)^2} \right),$$

f has a critical point at $y = 2/3$, and since f goes to infinity as y approaches 0 or 1, it follows that $f(2/3) = \frac{9}{16}\pi^2$ is the best upper bound.

5. Let Ω be the ellipse given in problem 2. This time consider the eigenvalue problem with Neumann boundary conditions,

$$\begin{cases} -\Delta u = \lambda u & (x,y) \in \Omega \\ \frac{\partial u}{\partial \nu} = 0 & (x,y) \in \partial\Omega. \end{cases} \tag{4}$$

(a) Compute the Rayleigh quotient of $w(x,y) = y$.

Answer. Using the same change of variables as above, we find

$$\begin{aligned} \|w\|_{L^2}^2 &= \int_0^{2\pi} \int_0^1 (4r^2 \sin^2\theta) 2r \, dr \, d\theta \\ &= \pi \int_0^1 8r^3 \, dr \\ &= 2\pi \end{aligned}$$

Next, since $\nabla w = (0,1)$,

$$\|\nabla w\|_{L^2}^2 = \iint_\Omega 1 \, dA = 2\pi$$

and thus the Rayleigh quotient for w is

$$\frac{\|\nabla w\|_{L^2}^2}{\|w\|_{L^2}^2} = 1.$$

(b) Prove that the Rayleigh quotient of w is a strict upper bound for the second eigenvalue of (4).

Answer. First recall that the first Neumann eigenvalue is zero, with the constant functions as the corresponding eigenfunctions. Since w is an odd function of y and the domain Ω is symmetric in y, we have

$$\langle w, C \rangle = C \iint_\Omega w \, dA = 0,$$

for any constant C, so w is orthogonal to the constant functions. Thus by the Minimum Principle for the second Neumann eigenvalue,

$$\tilde{\lambda}_2(\Omega) \leq \frac{\|\nabla w\|^2_{L^2}}{\|w\|^2_{L^2}}.$$

Now if the inequality above were actually an equality, then the second part of the Minimum Principle implies that w would be a solution of the eigenvalue problem (4) with eigenvalue $\tilde{\lambda}_2(\Omega)$. But w does not satisfy the Neumann boundary condition since, for instance, $\frac{\partial w}{\partial \nu}(0,2) = 1$. Hence the inequality is strict.

6. (a) Show that there does not exist a smooth function $f(x)$ with $f(0) = f(3) = 0$ and $\int_0^3 f'(x)^2\,dx = 1$, $\int_0^3 f(x)^2\,dx = 2$.

Answer: We know

$$\lambda_1 = \min\left\{\frac{||\nabla w||^2}{||w||^2} : w \in C^2(0,3); w \not\equiv 0, w(0) = 0, w(3) = 0\right\}$$

is the first eigenvalue of

$$(*)\begin{cases} -w'' = \lambda w & x \in (0,3) \\ w(0) = 0 = w(3). \end{cases}$$

We know the eigenvalues of this problem are given by $\lambda_n = \left(\frac{n\pi}{3}\right)^2$ for $n = 1, 2, \ldots$. Therefore, the first eigenvalues is given by

$$\lambda_1 = \left(\frac{\pi}{3}\right)^2 \approx 1.0966 > \frac{1}{2}.$$

Therefore for all trial functions $f \in C^2, f(0) = 0 = f(3)$, their Rayleigh Quotients must be $\geq \lambda_1$. Therefore, there can be no function f with Rayleigh Quotient $\frac{1}{2}$.

(b) Find two linearly independent functions $f_1(x)$ and $f_2(x)$ satisfying $f_i(0) = f_i(3) = 0$ and $\int_0^3 f_i'(x)^2\,dx = 2$ and $\int_0^3 f_i(x)^2\,dx = 1$, $i = 1, 2$.

Answer: The eigenfunctions of (*) are given by

$$v_n(x) = B_n \sin\left(\frac{n\pi}{3}x\right).$$

For simplicity, we normalize them, in which case, we let

$$v_n(x) = \sqrt{\frac{2}{3}} \sin\left(\frac{n\pi}{3}x\right).$$

We know that the Rayleigh quotient for each of the eigenfunctions $v_n(x)$ is equal to the corresponding eigenvalue λ_n. Therefore,

$$\frac{\int_0^3 |v_1'|^2\,dx}{\int_0^3 |v_1|^2\,dx} = \int_0^3 |v_1'|^2\,dx = \left(\frac{\pi}{3}\right)^2 \approx 1.0966$$

$$\frac{\int_0^3 |v_2'|^2\,dx}{\int_0^3 |v_2|^2\,dx} = \int_0^3 |v_2'|^2\,dx = \left(\frac{2\pi}{3}\right)^2 \approx 4.3865$$

We want functions f whose Rayleigh quotient is 2. We look for a function as a linear combination of the first two eigenfunctions. That is, we look for a function $f(x) = \alpha v_1(x) + \beta v_2(x)$. Therefore, we want to find α, β such that

$$\begin{aligned}\frac{\int_0^3 |f'|^2\,dx}{\int_0^3 |f|^2\,dx} &= \frac{\int_0^3 \alpha^2|v_1'|^2 + 2\alpha\beta v_1' v_2' + \beta^2 |v_2'|^2\,dx}{\int_0^3 \alpha^2 v_1^2 + 2\alpha\beta v_1 v_2 + \beta^2 v_2^2\,dx} \\ &= \frac{\alpha^2(\pi/3)^2 + \beta^2(2\pi/3)^2}{\alpha^2+\beta^2} = 2,\end{aligned}$$

using the fact that v_1, v_2 are orthogonal and v_1', v_2' are orthogonal on $(0,3)$. Take α, β such that $\alpha^2+\beta^2 = 1$. Then we just need

$$\begin{aligned}\alpha^2\left(\frac{\pi}{3}\right)^2 + \beta^2\left(\frac{2\pi}{3}\right)^2 &= 2 \\ \implies (1-\beta^2\left(\frac{\pi}{3}\right)^2 + \beta^2\left(\frac{2\pi}{3}\right)^2 &= 2.\end{aligned}$$

Our solutions of this are given by

$$\begin{aligned}\beta_\pm &= \pm\sqrt{\frac{18-\pi^2}{3\pi^2}} \\ \alpha_\pm &= \pm\sqrt{1-\frac{18-\pi^2}{3\pi^2}}.\end{aligned}$$

Therefore, two linearly independent solutions are given by

$$\begin{aligned}f_1(x) &= \alpha_+ v_1(x) + \beta_+ v_2(x) \\ f_2(x) &= \alpha_+ v_1(x) + \beta_- v_2(x).\end{aligned}$$