Math 220B - Summer 2003
Homework 6 Solutions

1. Consider the Neumann problem,

$$\begin{cases} -\Delta u = f & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

Assume the compatibility condition holds. That is,

$$-\int_\Omega f(x)\,dx = \int_{\partial\Omega} g(x)\,dS(x).$$

Just as the Green's function allowed us to find a representation formula for solutions to Poisson's equation on a bounded domain Ω, here we construct a *Neumann function* to derive a representation formula for the Neumann problem. Let $N(x,y)$ be defined as follows. Let

$$N(x,y) = \Phi(y-x) - \widetilde{h}^x(y) \qquad \forall y \in \overline{\Omega}$$

where $\widetilde{h}^x(y)$ is a solution of

$$\begin{cases} \Delta_y \widetilde{h}^x(y) = 0 & \forall y \in \Omega \\ \frac{\partial \widetilde{h}^x}{\partial \nu}(y) = \frac{\partial \Phi}{\partial \nu}(y-x) - C & \forall y \in \partial\Omega \end{cases}$$

for some appropriately chosen constant C. (In part (b), you will determine the necessary constant for a given region Ω. For now, you may assume C is arbitrary.)

(a) Use $N(x,y)$ to write a solution formula for

$$\begin{cases} -\Delta u = f & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

in terms of f, g, and N. (*Note: As we know, Poisson's equation with Neumann boundary conditions is only unique up to constants. Therefore, adding any constant to your solution formula will also give you a solution.*)

Answer: From our work in class, we know that for any $u \in C^2(\Omega)$, u has the following representation,

$$u(x) = -\int_\Omega \Delta u \Phi(x-y)\,dy + \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\Phi(x-y)\,dS(y) - \int_{\partial\Omega} u \frac{\partial \Phi}{\partial \nu}(x-y)\,dS(y).$$

If $\widetilde{h}^x$ is any smooth function on Ω, we know from lecture that

$$\int_\Omega \Delta_y \widetilde{h}^x(y) u(y)\,dy = \int_\Omega \Delta u \widetilde{h}^x(y)\,dy - \int_{\partial\Omega} \frac{\partial u}{\partial \nu}\widetilde{h}^x\,dS(y) + \int_{\partial\Omega} u \frac{\partial \widetilde{h}^x}{\partial \nu}\,dS(y).$$

Now assuming that $\widetilde{h}^x$ is a solution of the boundary-value problem for each $x \in \Omega$, we see that

$$0 = \int_\Omega \Delta u \widetilde{h}^x \, dy - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} \widetilde{h}^x \, dS(y) + \int_{\partial\Omega} u \left[\frac{\partial \Phi}{\partial \nu}(x-y) - C \right] dS(y).$$

Adding this equation to the first equation above, we have

$$u(x) = -\int_\Omega \Delta u [\Phi(x-y) - \widetilde{h}^x(y)] \, dy + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} [\Phi(x-y) - \widetilde{h}^x(y)] \, dS(y) - C \int_{\partial\Omega} u \, dS(y).$$

By definition of the Neumann function $N(x,y)$, we have

$$u(x) = -\int_\Omega \Delta u N(x,y) \, dy + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} N(x,y) \, dS(y) - C \int_{\partial\Omega} dS(y).$$

Therefore, if u is a solution of Poisson's equation on a bounded domain Ω with Neumann boundary conditions, then u may be written as

$$\boxed{u(x) = \int_\Omega N(x,y) f(y) \, dy + \int_{\partial\Omega} g(y) N(x,y) \, dS(y) - C \int_{\partial\Omega} u \, dS(y).}$$

(b) In the definition of $\widetilde{h}^x$, what must the constant C be? Explain.

Answer: Using the above representation formula, let $u \equiv 1$ on the closed, bounded domain $\overline{\Omega}$. Therefore, $\Delta u = 0$, $\partial u / \partial \nu = 0$ and $u = 1$ on the boundary. Therefore, by the above representation formula, we have

$$u(x) = -C \int_{\partial\Omega} dS(y).$$

Therefore,

$$\boxed{C = -\frac{1}{\int_{\partial\Omega} dS(y)}.}$$

2. (a) Find the Neumann function for $\mathbb{R}^n_+$.

Answer: In the case of $\Omega = \mathbb{R}^n_+$, $C = 0$. Therefore, to find the Neumann function $N(x,y)$, we need to find a corrector function $\widetilde{h}^x(y)$ for each $x \in \mathbb{R}^n_+$ such that

$$\begin{cases} \Delta_y \widetilde{h}^x(y) = 0 & \forall y \in \mathbb{R}^n_+ \\ \frac{\partial \widetilde{h}^x}{\partial \nu}(y) = \frac{\partial \Phi}{\partial \nu}(y-x) & \forall y \in \partial\mathbb{R}^n_+. \end{cases}$$

Now on $\partial\mathbb{R}^n_+$, $\frac{\partial \Phi}{\partial \nu}(y-x) = -\frac{\partial \Phi}{\partial y_n}(y-x)$. As we know,

$$\frac{\partial \Phi}{\partial y_n}(y-x) = \frac{x_n - y_n}{n\alpha(n)|y-x|^n}.$$

For $y \in \partial\mathbb{R}^n$, $y_n = 0$. Therefore,

$$-\frac{\partial \Phi}{\partial y_n}(y-x) = \frac{-x_n}{n\alpha(n)|y-x|^n}.$$

We know that $\Phi(y - x^*)$ is harmonic in y in $\mathbb{R}^n_+$ as long as $x^* \notin \mathbb{R}^n_+$. So, we would like to choose our corrector function $\widetilde{h}^x(y) = \Phi(y - x^*)$ for some x^*. Using the ideas for the Green's function, we let $x^* = (x_1, \ldots, x_{n-1}, -x_n)$ (the reflection point of x). In order to satisfy our boundary condition, we need to define $\widetilde{h}^x(y)$ as follows. Let

$$\widetilde{h}^x(y) = -\Phi(y - x^*).$$

Therefore, $\widetilde{h}^x$ is harmonic in y for all $y \in \mathbb{R}^n_+$, and $\frac{\partial h^x}{\partial \nu} = \frac{x_n}{n\alpha(n)|y-x^*|^n}$. Therefore, $\widetilde{h}^x$ defined above is the corrector function, and consequently, the Neumann function

$$\boxed{N(x, y) = \Phi(y - x) + \Phi(y - x^*).}$$

(b) Use the Neumann function for $\mathbb{R}^n_+$ to find the solution formula for

$$\begin{cases} \Delta u = 0 & x \in \mathbb{R}^n_+ \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\mathbb{R}^n_+. \end{cases}$$

Answer: Using the representation formula from the previous problem, we see that u is given by

$$\boxed{u(x) = \int_{\partial\mathbb{R}^n_+} g(y)[\Phi(y - x) + \Phi(y - x^*)]\, dS(y).}$$

3. Let Ω be an open, bounded subset of $\mathbb{R}^n$ with C^2 boundary. Let h be a continuous function on $\partial\Omega$. Let Φ be the fundamental solution of Laplace's equation on $\mathbb{R}^n$. Define the single-layer potential with moment h as

$$\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(y - x)\, dS(y).$$

(a) Show that $\overline{u}$ is defined and continuous for all $x \in \mathbb{R}^n$.

Answer: First, for $x \notin \partial\Omega$, $\Phi(x - y)$ is smooth, and, $\partial\Omega$ is a closed, bounded set. Therefore, $\overline{u}(x)$ is clearly defined.

Now, we consider $x \in \partial\Omega$.

Consider the case $n = 2$.

$$\begin{aligned} |\overline{u}(x)| &= \left| \frac{1}{2\pi} \int_{\partial\Omega} h(y) \ln|x - y|\, dS(y) \right| \\ &\le C|h(y)|_{L^\infty} \left| \int_{\partial\Omega} \ln|x - y|\, dS(y) \right|. \end{aligned}$$

Now away from the singularity, clearly that part of the integral is finite. Therefore, we just need to show that

$$(*) \qquad \int_{B(x,\delta)\cap\partial\Omega} \ln|x - y|\, dS(y)$$

is finite. Without loss of generality, we may assume $x = 0$. In addition, using the fact that $\partial\Omega$ is C^2, we have a function $f : \mathbb{R} \to \mathbb{R}$ such that $B(x, \delta) \cap \partial\Omega = \{(x_1, f(x_1))\}$ (assuming δ is sufficiently small). Therefore, (*) can be written as

$$\int_{-\delta}^{\delta} \ln|(y_1, f(y_1))| \sqrt{1 + |f'(y_1)|^2}\, dy_1.$$

But f is a C^2 function implies f' is a C^1 function, which implies $f'(y_1) = f'(0) + f''(C)y_1$. Therefore, $\sqrt{1 + |f'(y_1)|^2} \leq \sqrt{1 + |y_1|^2}$. Consequently, we have

$$\begin{aligned}\int_{-\delta}^{\delta} \ln|(y_1, f(y_1))| \sqrt{1 + |f'(y_1)|^2}\, dy_1 &\leq \int_{-\delta}^{\delta} \ln|(y_1, f(y_1))| \sqrt{1 + |y_1|^2}\, dy_1 \\ &\leq C \int_{-\delta}^{\delta} |(y_1, f(y_1))|^{-\epsilon} \sqrt{1 + |y_1|^2}\, dy_1,\end{aligned}$$

for any $\epsilon > 0$. But,

$$\begin{aligned}C \int_{-\delta}^{\delta} |(y_1, f(y_1))|^{-\epsilon} \sqrt{1 + |y_1|^2}\, dy_1 &\leq C \int_{-\delta}^{\delta} |y_1|^{-\epsilon} \sqrt{1 + |y_1|^2}\, dy_1 \\ &\leq C \int_{-\delta}^{\delta} |y_1|^{-\epsilon}\, dy_1 \leq C,\end{aligned}$$

as long as $\epsilon < 1$.
Therefore, $\overline{u}(x)$ is defined for all $x \in \Omega \subset \mathbb{R}^2$.
Next, we look at $n \geq 3$. Then

$$\begin{aligned}|\overline{u}(x)| &= \left| \frac{1}{n(n-2)\alpha(n)} \int_{\partial\Omega} \frac{h(y)}{|x - y|^{n-2}}\, dS(y) \right| \\ &\leq C |h(y)|_{L^\infty(\Omega)} \int_{\partial\Omega} \frac{1}{|x - y|^{n-2}}\, dS(y),\end{aligned}$$

using the fact that $\partial\Omega$ is an $n - 1$-dimensional surface in $\mathbb{R}^n$.
It remains only to show that $\overline{u}(x)$ is continuous. Clearly, for $x \in \Omega$ or $x \in \mathbb{R}^n \backslash \overline{\Omega}$, $\overline{u}(x)$ is continuous, because $\Phi(x - y)$ is smooth. Therefore, we only need to consider the case when $x \in \partial\Omega$.
Consider $x_0 \in \partial\Omega$. We need to show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $|u(x) - u(x_0)| < \epsilon$ for $|x - x_0| < \delta$. Let $B(x_0, \gamma)$ be a ball of radius γ about x_0. Let $B_\gamma \equiv \partial\Omega \cap B(x_0, \gamma)$. Let $A \equiv \partial\Omega \backslash \{\partial\Omega \cap B(x_0, \gamma)\}$. Write

$$\begin{aligned}\overline{u}(x) - \overline{u}(x_0) &= -\int_{\partial\Omega} h(y)[\Phi(x - y) - \Phi(x_0 - y)]\, dS(y) \\ &= -\int_{B_\gamma} h(y)[\Phi(x - y) - \Phi(x_0 - y)]\, dS(y) \\ &\quad - \int_A h(y)[\Phi(x - y) - \Phi(x_0 - y)]\, dS(y).\end{aligned}$$

As shown above, $\overline{u}(x)$ is defined for all $x \in \mathbb{R}^n$. Therefore, the first term is defined. We claim that it can be made arbitrarily small by choosing γ arbitrarily small. In particular,

$$\left| \int_{B_\gamma} h(y)[\Phi(x-y) - \Phi(x_0 - y)]\, dS(y) \right| \leq \int_{B_\gamma} h(y)\Phi(x-y)\, dS(y) + \int_{B_\gamma} h(y)\Phi(x_0 - y)\, dS(y).$$

Now for $x \notin \partial\Omega$, $\Phi(x-y)$ is bounded, and, therefore we have

$$\int_{B_\gamma} h(y)\Phi(x-y)\, dS(y) \leq C \int_{B_\gamma} dS(y) \leq \epsilon$$

by choosing γ sufficiently small. Now for $x \in \partial\Omega$, we use the fact that $\partial\Omega$ is C^2, and, therefore, can be written as a C^2 function locally. Without loss of generality, we may assume $x = 0$. There exists a function $f : \mathbb{R}^{n-1} \to \mathbb{R}$ and some $r > 0$ such that $\partial\Omega \cap B(0,r) \equiv \{y = (y_1, \ldots, y_{n-1}, f(y_1, \ldots, y_{n-1}))\}$. Therefore, letting $\widetilde{y} = (y_1, \ldots, y_{n-1})$, we have

$$\begin{aligned} \left| \int_{B_\gamma} h(y)\Phi(x-y)\, dS(y) \right| &\leq |h(y)|_{L^\infty(B_\gamma)} \int_{B_\gamma} |\Phi(y)|\, dS(y) \\ &\leq C \int_{\widetilde{B}(0,\gamma)} |\Phi((\widetilde{y}, f(\widetilde{y})))| \sqrt{1 + |\nabla f(\widetilde{y})|^2}\, d\widetilde{y}, \end{aligned}$$

where $\widetilde{B}(0,\gamma)$ is the ball of radius γ in $\mathbb{R}^{n-1}$. Using the fact that f is a C^2 function, we have $|\nabla f| \leq C$. But,

$$\int_{\widetilde{B}(0,\gamma)} |\Phi((\widetilde{y}, f(\widetilde{y})))|\, d\widetilde{y} = O(\gamma)$$

can be made arbitrarily small by choosing γ sufficiently small. I.e. in dimensions $n \geq 3$, we have

$$\begin{aligned} \int_{\widetilde{B}(0,\gamma)} |\Phi((\widetilde{y}, f(\widetilde{y})))|\, d\widetilde{y} &= C \int_0^\gamma \int_{\partial\widetilde{B}(0,r)} \frac{1}{|(\widetilde{y}, f(\widetilde{y}))|^{n-2}}\, dS(\widetilde{y})\, dr \\ &= C \int_0^\gamma \int_{\partial\widetilde{B}(0,r)} \frac{1}{(r^2 + f(\widetilde{y})^2)^{(n-2)/2}}\, dS(\widetilde{y})\, dr \\ &\leq C \int_0^\gamma dr = C\gamma. \end{aligned}$$

Then for γ chosen appropriately small, we can make the second term small by choosing $\delta \leq \gamma$ appropriately small and using the fact that $\Phi(x-y) - \Phi(x_0 - y)$ is uniformly continuous in y. We have

$$\begin{aligned} \left| \int_A h(y)[\Phi(x-y) - \Phi(x_0 - y)]\, dS(y) \right| &\leq C|\Phi(x-y) - \Phi(x_0 - y)|_{L^\infty(A)} \\ &\leq \frac{\epsilon}{2} \end{aligned}$$

for $|x - x_0| \leq \delta \leq \gamma$ where δ is chosen appropriately small.

(b) Show that $\Delta \overline{u}(x) = 0$ for $x \notin \partial\Omega$.

Answer: $\Phi(x-y)$ is smooth for $x \neq y$, and as discussed above, $\overline{u}(x)$ is defined for all $x \in \mathbb{R}^n$. Therefore, for $x \notin \partial\Omega$,

$$\begin{aligned}\Delta_x \overline{u}(x) &= -\Delta_x \int_{\partial\Omega} h(y)\Phi(x-y)\,dS(y) \\ &= -\int_{\partial\Omega} h(y)\Delta_x \Phi(x-y)\,dS(y) = 0.\end{aligned}$$

4. Let Ω be an open, bounded set in $\mathbb{R}^n$ with smooth boundary. Let $\Omega^c \equiv \mathbb{R}^n \backslash \overline{\Omega}$. Consider the exterior Neumann problem,

$$(*)\begin{cases} \Delta u = 0 & x \in \Omega^c \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega^c. \end{cases}$$

Assume g satisfies the condition,

$$\int_{\partial\Omega} g(x)\,dS(x) = 0. \qquad (**)$$

(*Note: Recall: This is not a necessary condition for solvability of the exterior Neumann problem.*) Suppose a solution u of (*) is given by the single-layer potential,

$$u(x) \equiv -\int_{\partial\Omega} h(y)\Phi(x-y)\,dS(y)$$

where h satisfies the integral equation

$$g(x) = \frac{1}{2}h(x) - \int_{\partial\Omega} h(y)\frac{\partial\Phi(x-y)}{\partial\nu_x}\,dS(y).$$

(a) Show that if g satisfies the condition (**), then

$$\int_{\partial\Omega} h(y)\,dS(y) = 0.$$

Answer: We integrate the integral equation for h with over $\partial\Omega$. In particular, we get

$$\begin{aligned}0 &= \int_{\partial\Omega} g(x)\,dS(x) \\ &= \frac{1}{2}\int_{\partial\Omega} h(x)\,dS(x) - \int_{\partial\Omega}\int_{\partial\Omega} h(y)\frac{\partial\Phi(x-y)}{\partial\nu_x}\,dS(y)\,dS(x) \\ &= \frac{1}{2}\int_{\partial\Omega} h(x)\,dS(x) + \int_{\partial\Omega} h(y)\left[-\int_{\partial\Omega}\frac{\partial\Phi(x-y)}{\partial\nu_x}\,dS(x)\right]dS(y) \\ &= \frac{1}{2}\int_{\partial\Omega} h(x)\,dS(x) + \int_{\partial\Omega} h(y)\frac{1}{2}\,dS(y) \\ &= \int_{\partial\Omega} h(x)\,dS(x),\end{aligned}$$

where we have used Gauss' Lemma which states that for $y \in \partial\Omega$,

$$-\int_{\partial\Omega} \frac{\partial\Phi(x-y)}{\partial\nu_x}\, dS(x) = \frac{1}{2}.$$

(b) Show that the solution u will have decay rate $O(|x|^{1-n})$ In particular, show $|u(x)| \le C|x|^{1-n}$. *Hint: By (a), write* $u(x) = -\int_{\partial\Omega} h(y)[\Phi(x-y) - \Phi(x)]\, dS(y)$.

Answer: If $g(x)$ satisfies the extra condition (*) above, then from (a), we know

$$\int_{\partial\Omega} h(y)\, dS(y) = 0,$$

and, therefore, we can write

$$u(x) = \int_{\partial\Omega} h(y)\Phi(x-y)\, dS(y) = \int_{\partial\Omega} h(y)[\Phi(x-y) - \Phi(x)]\, dS(y).$$

By the mean value theorem, there exists a point x^* on the line segment between $x-y$ and x such that

$$\Phi(x-y) - \Phi(x) = \nabla\Phi(x^*) \cdot (-y).$$

By calculating $\nabla\Phi(x)$, we see that

$$\nabla\Phi(x^*) = \frac{C}{|x^*|^{n-1}} = O(|x|^{1-n}),$$

using the fact that x^* is between $x-y$ and x. Therefore,

$$\begin{aligned}
|u(x)| &\le \int_{\partial\Omega} |h(y)||\Phi(x-y) - \Phi(x)|\, dS(y) \\
&\le |h(y)|_{L^\infty} \int_{\partial\Omega} |\nabla\Phi(x^*)||y|\, dS(y) \\
&\le C|x|^{1-n}.
\end{aligned}$$

This gives us a decay rate $O(|x|^{1-n})$.

5. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $\Omega^c \equiv \mathbb{R}^n \backslash \overline{\Omega}$. Prove there exists at most one solution u which decays to 0 as $|x| \to +\infty$ of the following

$$\begin{cases} \Delta u = f & x \in \Omega^c \\ u = g & x \in \partial\Omega. \end{cases}$$

Answer: Suppose there exist two solutions u and v. Define the set $\Omega_R^c \equiv \Omega^c \cap B(0,R)$. Let $w = u - v$. Now using the fact that $|u|, |v| \to 0$ as $|x| \to +\infty$, we see that for all $\epsilon > 0$ there exists an $R > 0$ such that $|w(x)| < \epsilon$ if $|x| > R$. Let $\epsilon > 0$ Fix R such that $|w(x)| < \epsilon$ if $|x| \ge R$. Then w is a solution of

$$\begin{cases} \Delta w = 0 & x \in \Omega_R^c \\ w = 0 & x \in \partial\Omega \\ |w| < \epsilon & x \in \partial B(0,R). \end{cases}$$

Therefore, by the maximum principle for harmonic functions,

$$\max_{\overline{\Omega}_R^c} w = \max_{\partial\Omega_R^c} w < \epsilon.$$

Similarly, defining $\widetilde{w} = v - u$, we conclude that

$$\max_{\overline{\Omega}_R^c} \widetilde{w} < \epsilon.$$

Therefore, we conclude that $|u - v| < \epsilon$ on $\overline{\Omega}_R^c$. Since this is true for all ϵ by choosing R sufficiently large, we conclude that $u = v$.