Math 220B - Summer 2003
Homework 6
Due Thursday, August 7, 2003

1. Consider the Neumann problem,

$$\begin{cases} -\Delta u = f & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

Assume the compatibility condition holds. That is,

$$-\int_{\Omega} f(x)\,dx = \int_{\partial\Omega} g(x)\,dS(x).$$

Just as the Green's function allowed us to find a representation formula for solutions to Poisson's equation on a bounded domain Ω, here we construct a *Neumann function* to derive a representation formula for the Neumann problem. Let $N(x,y)$ be defined as follows. Let

$$N(x,y) = \Phi(y-x) - \widetilde{h}^x(y) \qquad \forall y \in \overline{\Omega}$$

where $\widetilde{h}^x(y)$ is a solution of

$$\begin{cases} \Delta_y \widetilde{h}^x(y) = 0 & \forall y \in \Omega \\ \frac{\partial \widetilde{h}^x}{\partial \nu}(y) = \frac{\partial \Phi}{\partial \nu}(y-x) - C & \forall y \in \partial\Omega \end{cases}$$

for some appropriately chosen constant C. (In part (b), you will determine the necessary constant for a given region Ω. For now, you may assume C is arbitrary.)

(a) Use $N(x,y)$ to write a solution formula for

$$\begin{cases} -\Delta u = f & x \in \Omega \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega \end{cases}$$

in terms of f, g, and N. (*Note: As we know, Poisson's equation with Neumann boundary conditions is only unique up to constants. Therefore, adding any constant to your solution formula will also give you a solution.*)

(b) In the definition of $\widetilde{h}^x$, what must the constant C be? Explain.

2. (a) Find the Neumann function for $\mathbb{R}^n_+$.

(b) Use the Neumann function for $\mathbb{R}^n_+$ to find the solution formula for

$$\begin{cases} \Delta u = 0 & x \in \mathbb{R}^n_+ \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\mathbb{R}^n_+. \end{cases}$$

3. Let Ω be an open, bounded subset of $\mathbb{R}^n$ with C^2 boundary. Let h be a continuous function on $\partial\Omega$. Let Φ be the fundamental solution of Laplace's equation on $\mathbb{R}^n$. Define the single-layer potential with moment h as

$$\overline{u}(x) = -\int_{\partial\Omega} h(y)\Phi(y-x)\,dS(y).$$

   (a) Show that $\overline{u}$ is defined and continuous for all $x \in \mathbb{R}^n$.

   (b) Show that $\Delta\overline{u}(x) = 0$ for $x \notin \partial\Omega$.

4. Let Ω be an open, bounded set in $\mathbb{R}^n$ with smooth boundary. Let $\Omega^c \equiv \mathbb{R}^n \backslash \overline{\Omega}$. Consider the exterior Neumann problem,

$$(*) \begin{cases} \Delta u = 0 & x \in \Omega^c \\ \frac{\partial u}{\partial \nu} = g & x \in \partial\Omega^c. \end{cases}$$

Assume g satisfies the condition,

$$\int_{\partial\Omega} g(x)\,dS(x) = 0. \qquad (**)$$

(*Note: Recall: This is not a necessary condition for solvability of the exterior Neumann problem.*) Suppose a solution u of (*) is given by the single-layer potential,

$$u(x) \equiv -\int_{\partial\Omega} h(y)\Phi(x-y)\,dS(y)$$

where h satisfies the integral equation

$$g(x) = \frac{1}{2}h(x) - \int_{\partial\Omega} h(y)\frac{\partial\Phi(x-y)}{\partial\nu_x}\,dS(y).$$

   (a) Show that if g satisfies the condition (**), then

$$\int_{\partial\Omega} h(y)\,dS(y) = 0.$$

   (b) Show that the solution u will have decay rate $O(|x|^{1-n})$ In particular, show $|u(x)| \le C|x|^{1-n}$. *Hint: By (a), write* $u(x) = -\int_{\partial\Omega} h(y)[\Phi(x-y) - \Phi(x)]\,dS(y)$.

5. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $\Omega^c \equiv \mathbb{R}^n \backslash \overline{\Omega}$. Prove there exists at most one solution u which decays to 0 as $|x| \to +\infty$ of the following

$$\begin{cases} \Delta u = f & x \in \Omega^c \\ u = g & x \in \partial\Omega. \end{cases}$$