Math 220B - Summer 2003
Homework 5
Due Thursday, July 31, 2003

1. Let $\Omega = (0,k) \times (0,l)$. Use separation of variables to solve the following boundary-value problem for Laplace's equation on a square,

$$\begin{cases} \Delta u = 0 & (x,y) \in \Omega \\ u(0,y) = 0, u_x(k,y) = \phi(y) & 0 < y < l \\ u_y(x,0) = 0, u(x,l) = 0 & 0 < x < k. \end{cases}$$

2. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Prove uniqueness of solutions of

$$\begin{cases} \Delta u = f & x \in \Omega \\ \frac{\partial u}{\partial \nu} + \alpha u = g & x \in \partial\Omega \end{cases}$$

for $\alpha > 0$.

3. Let $\Omega \equiv \{(x,y) : a^2 < x^2 + y^2 < b^2\}$ be an annular region in $\mathbb{R}^2$. Consider

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ \frac{du}{d\nu} + \alpha u = g(\theta) & x^2 + y^2 = a^2 \\ \frac{du}{d\nu} + \beta u = h(\theta) & x^2 + y^2 = b^2 \end{cases}$$

where ν is the outer unit normal to Ω.

   (a) Solve this boundary-value problem in the case when $\alpha = \beta = 1$, $a = 1$, $b = 2$, $h(\theta) = 0$ and $g(\theta)$ is an arbitrary function.

   (b) From the result from the previous problem, we know the solution to part (a) is unique. Prove that uniqueness may fail if either α or β are negative, by finding two solutions of

$$\begin{cases} u_{xx} + u_{yy} = 0 & (x,y) \in \Omega \\ \frac{du}{d\nu} + 2u = 0 & x^2 + y^2 = 1 \\ \frac{du}{d\nu} - u = 0 & x^2 + y^2 = 4. \end{cases}$$

4. (a) Find the one-dimensional Green's function for $\Omega = (0,l)$, That is, find the function $G(x,y)$ such that for each $x \in \Omega$,

$$\begin{cases} -\Delta_y G(x,y) = \delta_x & y \in \Omega \\ G(x,y) = 0 & y \in \partial\Omega. \end{cases}$$

   You may use the fact that the fundamental solution of Laplace's equation in one dimension is $\Phi(x) = -\frac{1}{2}|x|$.

   (b) Use the Green's function above to solve the ODE

$$\begin{cases} u''(x) = 1 & x \in (0,1) \\ u(0) = 3 \\ u(1) = 2. \end{cases}$$

5. Find the Green's function for Laplace's equation on the half-ball $\Omega \equiv \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 < 1, z > 0\}$.

6. Find the Green's function for Laplace's equation in the wedge $\Omega = \{(x_1,x_2) \in \mathbb{R}^2, x_2 > 0, x_1 > x_2\}$.