Math 220B - Summer 2003
Homework 4
Due Tuesday, July 22, 2003

1. Let Ω be an open, bounded set in $\mathbb{R}^n$. Let $\phi \in C(\overline{\Omega})$, $g \in C(\partial\Omega)$, and suppose $u \in C^2(\overline{\Omega} \times [0,\infty))$ is a solution of

$$(1) \quad \begin{cases} u_t(x,t) = k\Delta u(x,t) & (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = g(x) & (x,t) \in \partial\Omega \times [0,\infty) \\ u(x,0) = \phi(x) & x \in \Omega. \end{cases}$$

Also suppose that $v \in C^2(\overline{\Omega})$ is a solution of

$$(2) \quad \begin{cases} \Delta v(x) = 0 & x \in \Omega \\ v(x) = g(x) & x \in \partial\Omega. \end{cases}$$

Prove $u(x,t) \to v(x)$ in $L^2(\Omega)$ as $t \to \infty$ as follows.

(a) For Ω an open, bounded set in $\mathbb{R}^n$, there exists a constant C (depending only on Ω) such that for every $f \in C^1(\overline{\Omega})$ with $f(x) = 0$ for all $x \in \partial\Omega$,

$$(*) \quad \|f\|_{L^2(\Omega)} \leq C\|\nabla f\||_{L^2(\Omega)}.$$

Prove this inequality for the case when $\Omega = (a,b) \subset \mathbb{R}$.

(b) Using (*), prove that for u and v solutions of (1) and (2) respectively, $u(x,t) \to v(x)$ in $L^2(\Omega)$ as $t \to +\infty$. (*Hint: Let $w(x,t) \equiv u(x,t) - v(x)$. Consider the PDE that w solves. Show that $||w(x,t)||^2_{L^2(\Omega)} \to 0$ as $t \to +\infty$.*)

2. Let $B_n(0,a)$ be the unit ball in $\mathbb{R}^n$ centered at 0 with radius $a > 0$.

(a) Let $\alpha > 0$. Show that

$$\int_{B_n(0,a)} \frac{1}{|x|^\alpha}\, dx < \infty$$

if and only if $\alpha < n$. In particular, evaluate the integral for $n > \alpha > 0$.

(b) Give conditions on α for which

$$\int_{\mathbb{R}^n \setminus B_n(0,a)} \frac{1}{|x|^\alpha}\, dx < \infty.$$

3. Find all radial solutions of

$$-\Delta u + u = 0 \qquad x \in \mathbb{R}^3.$$

4. Prove that

$$u(x) \equiv \frac{e^{-|x|}}{4\pi|x|}$$

satisfies

$$-\Delta u + u = \delta_0 \qquad x \in \mathbb{R}^3$$

in the sense of distributions.