Math 220B - Summer 2003
Homework 3
Due Thursday, July 17, 2003

1. Suppose $f_n : \mathbb{R} \to \mathbb{R}$ is a sequence of continuous, nonnegative functions such that

$$f_n(x) \to \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases}$$

as $n \to +\infty$. In addition, assume

$$\int_{-\infty}^{\infty} f_n(x)\, dx = 1$$

for all n, and there exists a closed, bounded subset $K \subset \mathbb{R}$ such that $f_n(x) \equiv 0$ for $x \notin K$. (That is, $\text{supp}(f_n) \subset K$ for all n.)

Show that

$$f_n \rightharpoonup \delta_0$$

as $n \to +\infty$ in the sense of distributions. (That is, prove weak convergence.)

2. (a) We define the Fourier transform of a distribution as follows. Let $F : \mathcal{D} \to \mathbb{R}$ be a distribution. Then $\widehat{F}$ is defined as the distribution such that $(\widehat{F}, \phi) = (F, \widehat{\phi})$ for all $\phi \in \mathcal{D}$. Using this definition, compute the Fourier transform of the delta function.

   (b) Use your answer to part (a) to solve

$$y'(x) - ay = \delta_0 \qquad a \in \mathbb{R}, a \neq 0.$$

3. Define

$$u(x) = \begin{cases} 0 & x < 0 \\ \sin x & x \geq 0 \end{cases}$$

Show that $u'' + u = \delta_0$ in the sense of distributions, where δ denotes the delta function.

4. Solve

$$\begin{cases} u_t - ku_{xx} + u = 0 & 0 < x < 1, t > 0 \\ u(x,0) = \phi(x) & \\ u_x(0,t) - u(0,t) = g(t) & \\ u_x(l,t) = h(t). & \end{cases}$$

5. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Use energy methods to prove uniqueness of solutions to

$$\begin{cases} u_t - k\Delta u + u = f & x \in \Omega, t > 0 \\ u(x,0) = \phi(x) & \\ \frac{\partial u}{\partial \nu} + au = g & x \in \partial\Omega \end{cases}$$

for $a \geq 0$.