Math 220B - Summer 2003
Homework 1
Due Thursday, July 3, 2003

1. Consider the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X \text{ satisfies symmetric B.C.s.} & x = 0, l. \end{cases}$$

   Suppose

$$f(x)f'(x)\big|_{x=a}^{x=b} \leq 0$$

   for all real-valued functions $f(x)$ which satisfy the boundary conditions. Show there are no negative eigenvalues.

2. Consider the eigenvalue problem,

$$\begin{cases} X'' = -\lambda X & a < x < b \\ X \text{ satisfies certain B.C.'s.} \end{cases}$$

   Suppose μ is an eigenvalue of multiplicity $m > 1$. Let $X_1, \ldots, X_m$ denote m linearly independent eigenfunctions (which may or may not be orthogonal) associated with the eigenvalue μ. Use these eigenfunctions to construct m eigenfunctions $Y_1, \ldots, Y_m$ which are necessarily orthogonal.

3. Consider the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X'(0) + X(0) = 0 \\ X(l) = 0. \end{cases}$$

   (a) Find an equation for the positive eigenvalues.

   (b) Show graphically that there are an infinite number of positive eigenvalues.

   (c) Show that $\lambda = 0$ is an eigenvalue if and only if $l = 1$. Find a corresponding eigenfunction in this case.

   (d) Show that if $l \leq 1$, then there are no negative eigenvalues, but if $l > 1$, then there is one negative eigenvalue. Find the corresponding eigenfunction.

4. Use separation of variables to solve

$$\begin{cases} u_t - ku_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \\ u_x(l,t) = 0. \end{cases}$$

5. Use separation of variables to solve

$$\begin{cases} u_t - ku_{xx} + u_x = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) \\ u(0,t) = 0 \\ u_x(0,t) = 0. \end{cases}$$

(*Hint:* Introduct a function $f(x)$ such that $v(x,t) = f(x)u(x,t)$ will satisfy a PDE of the form

$$v_t - kv_{xx} + av = 0$$

with new initial and boundary conditions. Solve the equation for v, and from this solution, solve for u.)