Math 220a - Fall 2002
Homework 4 Solutions

1. Classify the following equations as elliptic, parabolic, or hyperbolic.

   (a) $2u_{xx} + 2u_{xy} + 2u_{xz} + 3u_{yy} - 4u_{yz} + 3u_{zz} = 0$

   (b) $2u_{xz} + u_{yy} = 0$

   (c) $7u_{xx} - 10u_{xy} - 22u_{yz} + 7u_{yy} - 16u_{xz} - 5u_{zz} = 0$

   Answer:

   (a) We have:

   $$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & -2 \\ 1 & -2 & 3 \end{pmatrix}$$

   The eigenvalues are 0, 3, and 5. *parabolic*

   (b) We have :

   $$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

   The eigenvalues are 1,1,and -1. *hyperbolic*

   (c) we have:

   $$\begin{pmatrix} 7 & -5 & -8 \\ -5 & 7 & -11 \\ -8 & -11 & -5 \end{pmatrix}$$

   The eigenvalues are -15.4862, 10.9407, and 13.5455. *hyperbolic*

2. Show that every elliptic equation of the form

   $$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y),$$

   where $b^2 - 4ac < 0$ can be brought into the form

   $$\tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} + k\tilde{u} = F(\xi, \eta)$$

   through a change of variables. In particular, first give an appropriate linear change of variables to show the equation can be written in the form

   $$u_{\xi\xi} + u_{\eta\eta} + \alpha u_\xi + \beta u_\eta + \gamma u = h(\xi, \eta)$$

for some constants α, β, γ. Then introduce a change of variables for the dependent variable u to eliminate the first derivative terms, to show that the equation can be written in the form

$$\tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} + k\tilde{u} = F(\xi, \eta).$$

Answer: let

$$x = \sqrt{a}\xi;\ y = \frac{b}{2\sqrt{a}}\xi + \sqrt{(}c - b^2/4a)\eta$$

then we can get that

$$u_{\xi\xi} + u_{\eta\eta} + \alpha u_{\xi} + \beta u_{\eta} + \gamma u = h(\xi, \eta)$$

in which α, β and γ are some suitable number. Then, we let

$$\bar{u} = ue^{\frac{\alpha\xi+\beta\eta}{2}}$$

and

$$k = \gamma - \frac{\alpha^2 + \beta^2}{4},\ \ F = he^{\frac{\alpha\xi+\beta\eta}{2}}$$

then we get

$$\bar{u}_{\xi\xi} + \bar{u}_{\eta\eta} + k\bar{u} = F(\xi, \eta)$$

3. Reduce the following second-order equation to a system of first-order equations

$$u_{tt} - 4u_{xt} - 5u_{xx} = 0.$$

Then use the method of characteristics to derive the general solution.

Answer:Since

$$u_{tt} - 4u_{xt} - 5u_{xx} = (\partial t - 5\partial x)(\partial t + \partial x)u = 0$$

therefore, we let

$$v = (\partial t + \partial x)u$$

hence

$$(\partial t - 5\partial x)v = 0$$

we get

$$v = g(x + 5t)$$

for some function g, then let's try to find our the solution of

$$u_t + u_x = g(x + 5t)$$

then we get

$$\begin{cases} \dfrac{dt}{ds} = 1 \\ \dfrac{dx}{ds} = 1 \\ \dfrac{du}{ds} = g(x + 5t) \end{cases}$$

One solution of this system is $t = s, x = 5s$ and $dz/ds = g(10s)$, which implies tthat

$$z(s) = \frac{1}{10}\int_0^{10s} g(\alpha)d\alpha$$

we arrive at a particular solution

$$u(x,t) = \frac{1}{10}\int_0^{x+5t} h(\alpha)d\alpha = f(x+5t)$$

We also note that any function of the form $u(x,t) = f(x-t)$ is a solution. So the general solution of the equation is

$$u(x,t) = f(x-t) + g(x+5t)$$

4. Consider the IVP

$$(*)\begin{cases} u_{tt} + u_{xt} - 12u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

(a) Make a change of variables to reduce the PDE to canonical form

$$u_{\xi\xi} - u_{\eta\eta} = 0.$$

Write the general solution of

$$u_{tt} + u_{xt} - 12u_{xx} = 0.$$

Answer: Let

$$t = \xi, \quad x = (\xi + 7\eta)/2$$

then we get

$$u_{\xi\xi} - u_{\eta\eta} = 0$$

so the general solution of the PDE is

$$u(x,t) = u(\xi,\eta) = f(\xi+\eta) + g(\xi-\eta) = f((2x+6t)/7) + g((2x-8t)/7)$$

(b) Solve the IVP (*).

Answer: Use the general solution we get in part (a) and plug in the initial condition, we get

$$\begin{cases} f(2x/7) + g(2x/7) = \phi(x) \\ \frac{6}{7}f^{'}(2x/7) - \frac{8}{7}g^{'}(2x/7) = \psi(x) \end{cases}$$

therefore

$$\begin{cases} f^{'}(x) = 2\phi^{'}(7x/2) + 1/2\psi^{'}(7x/2) \\ g^{'}(x) = 3/2\phi^{'}(7x/2) - 1/2\psi(7x/2) \end{cases}$$

by change of variables, we get

$$\begin{cases} f((2x+6t)/7) = \frac{4}{7}\int_0^{x+3t} \phi^{'}(u)du + \frac{1}{7}\int_0^{x+3t} \psi(u)du + f(0) \\ g((2x-8t)/7) = \frac{3}{7}\int_0^{x-4t} \phi^{'}(u)du + \frac{1}{7}\int_{x-4t}^{0} \psi(u)du + g(0) \end{cases}$$

so we get the solution

$$u(x,t) = \frac{4\phi(x+3t)+3\phi(x-4t)}{7} + \frac{1}{7}\int_{x-4t}^{x+3t} \psi(u)du$$

5. We say u is a weak solution of the wave equation,

$$\begin{cases} u_{tt} - u_{xx} = 0 \quad -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

if

$$\int_0^\infty \int_{-\infty}^{\infty} u[v_{tt} - v_{xx}]\, dx\, dt + \int_{-\infty}^{\infty} \phi(x) v_t(x,0)\, dx - \int_{-\infty}^{\infty} \psi(x) v(x,0)\, dx = 0$$

for all $v \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support. Let f be a piecewise continuous function with a jump at y_0. Show that $u(x,t) = f(x+t)$ is a weak solution of the wave equation. (*Note:* Similarly it can be shown that a piecewise continuous function of the form $g(x-t)$ is also a weak solution.)

Answer: Like we did in class, let

$$\Omega_1 = \{(x,t) : 0 < t < \infty,\ -\infty < x < y_0 - t\}$$

$$\Omega_2 = \{(x,t) : 0 < t < \infty,\ y_0 - t < x < \infty\}$$

Then we know

$$\int_0^\infty \int_{-\infty}^{\infty} u(v_{tt} - v_{xx}) dxdt = \int\int_{\Omega_1} u(v_{tt} - v_{xx}) dxdt + \int\int_{\Omega_2} u(v_{tt} - v_{xx} dxdt$$

and we know that

$$u(v_{tt} - v_{xx}) = (uv_t - u_t v)_t + (u_x v - uv_x)_x + v(u_{tt} - u_{xx})$$

in Ω_1 and Ω_2, $u_{tt} - u_{xx} = 0$, hence

$$\begin{aligned} \int\int_{\Omega_1} (\dots) dxdt &= \int\int_{\Omega_1} (u_x v - uv_x)_x dxdt + (uv_t - u_t v)_t dxdt \\ &= \int_0^\infty (\psi(y_0) v(y_0 - t, t) - f(y_0) v_x(y_0 - t, t)) dt \\ &\quad + \int_{-\infty}^{0} (f(y_0) v_t(x, y_0 - x) - \psi(y_0) v(x, y_0 - x)) dx \\ &\quad - \int_{-\infty}^{0} (u(x,0) v_t(x,0) - u_t(x,0) v(x,0)) dx \end{aligned}$$

And

$$\begin{aligned}
\int\int_{\Omega_2} (\dots) dxdt &= \int\int_{\Omega_2} (u_x v - u v_x)_x dxdt + (u v_t - u_t v) dxdt \\
&= \int_0^\infty (-\psi(y_0) v(y_0 - t, t) + f(y_0) v_x(y_0 - t, t)) dt \\
&- \int_{-\infty}^0 (f(y_0) v_t(x, y_0 - x) - \psi(y_0) v(x, y_0 - x)) dx \\
&- \int_0^\infty (u(x,0) v_t(x,0) - u_t(x,0) v(x,0)) dx
\end{aligned}$$

Add them up, we get

$$\int_0^\infty \int_{-\infty}^\infty u(v_t t - v_x x) dxdt = -\int_{-\infty}^\infty \phi(x) v_t(x,0) dx + \int_{-\infty}^\infty \psi(x) v(x,0) dx$$