Math 220a - Hw1 solutions (Revised)

1. Linear - (b)

   Semilinear - (a),(d)

   Quasilinear -(e)

   Fully nonlinear - (c)

2. The equations for the characteristic curves are

$$\begin{aligned} \frac{dt(r,s)}{ds} &= 1 \\ \frac{dx(r,s)}{ds} &= x \\ \frac{du(r,s)}{ds} &= t^3 \end{aligned}$$

   The boundary conditions can be re-written in terms of (s, r) as

$$\begin{aligned} t(r,0) &= 0 \\ x(r,0) &= r \\ u(r,0) &= \phi(r) \end{aligned}$$

   Integrating and using the boundary conditions,

$$\begin{aligned} t &= s + c_1(r) \\ &= s \qquad \text{Since } t(r,0) = 0 \\ x &= c_3(r)e^t \\ &= re^t \qquad \text{Since x(r,0)=r} \\ u &= s^4/4 + c_2(r) \\ &= \frac{s^4}{4} + \phi(r) \qquad \text{Since } u(r,0) = \phi(r) \end{aligned}$$

   Eliminating r, s form the above equations we get,

$$u(x,t) = \frac{t^4}{4} + \phi(xe^{-t})$$

3. The equations for the characteristic curves and boundary conditions in terms of r, s are

$$\begin{array}{ccc} \dfrac{dt(r,s)}{ds} = 1 & \frac{dx(r,s)}{ds} = x & \dfrac{du(r,s)}{ds} = u^3 \\ t(r,0) = 0 & x(r,0) = r & u(r,0) = \sin(r) \end{array}$$

Integration together with the boundary conditions yield,

$$
\begin{aligned}
t &= s \\
x &= re^{-t} \\
u^{-3}du &= ds \\
\text{(i.e.)} \quad \frac{u^-2}{-2} &= s + C(r) \\
\text{with } C(r) \text{ given by,} & \\
C(r) &= -\frac{1}{2sin^2(r)}
\end{aligned}
$$

Eliminating r, s form the above equations we get,

$$u(x,t) = \frac{\sin(xe^{-t})}{\sqrt{1 - 2t\sin^2(xe^{-t})}}$$

Now u blows up at a time $T > 0$ such that

$$\sin^2(xe^{-T}) = \frac{1}{2T}$$

As long as $T < 0.5$ the RHS is grater than 1 and cannot equal the LHS. When $T = 0.5$ the equation holds with $x = \frac{\pi}{2}e^{0.5}$. Hence $T = 0.5$, is the earliest time at which u blows up for some x.

4. The characteristic curves for the given PDE along the boundary (x axis) have the direction vector $(x, 1)$. This is tangential to the boundary at the origin. Therefore, if the boundary specification is consistent with the PDE at the origin we would have infinite solutions and if not we would have no solution.

   At the origin the PDE becomes $u_x = 0$. While the boundary condition specified by (a) is $u_x|_{(0,0)} = (\sin(x))_x|_{0,0} = cos(x)|_{0,0} = 1$. Thus we have inconsistency and hence no solution.

   While in the case of (b) we have $u_x|_{(0,0)} = (\cos(x))_x|_{0,0} = -sin(x)|_{0,0} = 0$, which is consistent with what the PDE specifies, hence we have infinite solutions.

5. Solve by characteristics:

$$\Gamma = (x(r,0), t(r,0)) = (r,0)$$

$$\begin{cases} \frac{dx}{ds} = z \\ \frac{dt}{ds} = 1 \\ \frac{dz}{ds} = 0 \end{cases}$$

   Since z is constant w.r.t. s,

$$\begin{cases} x = zs + r \\ t = s \\ z(r,s) = z(r,0) = \sin r \end{cases}$$

Solving for r,s, we obtain:

$$u(x,t) = \sin(x - ut)$$

To find u_x, differentiate the above w.r.t. x

$$u_x = \cos(x - ut)(1 - tu_x)$$

$$u_x = \frac{\cos(x - ut)}{1 + tcos(x - ut)}$$

A singularity occurs if $1 + t\cos r = 0$. In particular, when $T = t = 1$, and $x_0 = r = \pi + 2n\pi, n = 1, 2, ...$