Math 220A - Fall 2002
Homework 8
Due Friday, December 6, 2002

1. Consider
$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^3, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$
   Suppose ϕ, ψ are supported in the annular region $a < |x| < b$.

   (a) Find the time $T_1 > 0$ such that $u(x,t)$ is definitely zero for $t > T_1$ in the case when
      i. $|x| > b$
      ii. $a < |x| < b$
      iii. $|x| < a$.

   (b) Find the time $T_2 > 0$ such that $u(x,t)$ is definitely zero for $0 < t < T_2$ in the case when
      i. $|x| > b$
      ii. $|x| < a$.

   (c) Consider the same questions for $n = 2$ dimensions.

2. Solve
$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & (x,y,z) \in \mathbb{R}^3, t > 0 \\ u(x,y,z,0) = 1 \\ u_t(x,y,z,0) = x^2 + y^2 + z^2. \end{cases}$$

3. Solve
$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & (x,y) \in \mathbb{R}^2, t > 0 \\ u(x,y,0) = 0 \\ u_t(x,y,0) = x^2 + y^2 \end{cases}$$

4. Solve
$$\begin{cases} U_t + AU_x = 0 \\ U(x,0) = \Phi(x) \end{cases}$$
   where
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$
   and
$$\Phi(x) = \begin{bmatrix} \sin(x) \\ 1 \\ e^2 \end{bmatrix}.$$

5. Consider the symmetric hyperbolic system

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{x_1} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(a) Find the smallest ball in $\mathbb{R}^2$ in which the domain of dependence of $U(3, 4, 10)$ will lie. That is, find M such that the value of U at the point $(3, 4, 10)$ depends at most on the value of the initial data $U(x_1, x_2, 0)$ in the ball of radius $10M$ about $(3, 4)$.

(b) Show that the ball you found in part (a) is the smallest ball in which you can guarantee the domain of dependence will lie, by showing there exists a direction $\xi = (\xi_1, \xi_2)$, where $|\xi| = 1$ for which there exists a plane wave solution $U(x_1, x_2, t) = V(x \cdot \xi - Mt)$; that is, a plane wave solution which travels at speed M. You don't need to calculate the plane wave solution.

(c) Find two plane wave solutions which propagate in the direction $(\xi_1, \xi_2) = (3/5, 4/5)$; that is, find two general solutions of the form $V_1(\xi \cdot x - \sigma_1 t)$, $V_2(\xi \cdot x - \sigma_2 t)$.