6 Wave Equation on an Interval: Separation of Variables

6.1 Dirichlet Boundary Conditions

Ref: Strauss, Chapter 4

We now use the separation of variables technique to study the wave equation on a finite interval. As mentioned above, this technique is much more versatile. In particular, it can be used to study the wave equation in higher dimensions. We will discuss this later, but for now will continue to consider the one-dimensional case. We start by considering the wave equation on an interval with *Dirichlet boundary conditions*,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u(0,t) = 0 = u(l,t). \end{cases} \tag{6.1}$$

Our plan is to look for a solution of the form $u(x,t) = X(x)T(t)$. Suppose we can find a solution of this form. Plugging u into the wave equation above, we see that the functions X, T must satisfy

$$X(x)T''(t) = c^2 X''(x)T(t). \tag{6.2}$$

Dividing (6.2) by $-c^2XT$, we see that

$$-\frac{T''}{c^2T} = -\frac{X''}{X} = \lambda$$

for some constant λ, which implies

$$\begin{aligned} -X''(x) &= \lambda X(x) \\ -T''(t) &= \lambda c^2 T(t). \end{aligned}$$

Our boundary conditions $u(0,t) = 0 = u(l,t)$ imply $X(0)T(t) = 0 = X(l)T(t)$ for all t. Combining this boundary condition with the ODE for X, we see that X must satisfy

$$\begin{cases} -X''(x) = \lambda X(x) \\ X(0) = 0 = X(l). \end{cases} \tag{6.3}$$

If there exists a constant λ satisfying (6.3) for some function X, which is not identically zero, we say λ is an **eigenvalue** of $-\partial_x^2$ on $[0,l]$ subject to Dirichlet boundary conditions. The corresponding function X is called an **eigenfunction** of $-\partial_x^2$ on $[0,l]$ subject to Dirichlet boundary conditions.

Claim 1. *The eigenvalues of (6.3) are $\lambda_n = (n\pi/l)^2$ with corresponding eigenfunctions $X_n(x) = \sin(n\pi x/l)$.*

Proof. We need to look for all the eigenvalues of (6.3). First, we look for any positive eigenvalues. Suppose λ is a positive eigenvalue. Then $\lambda = \beta^2 > 0$. Therefore, we need to find solutions of

$$\begin{cases} X''(x) + \beta^2 X(x) = 0 \\ X(0) = 0 = X(l). \end{cases}$$

Solutions of this ODE are given by

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

for arbitrary constants C, D. The boundary condition

$$X(0) = 0 \implies C = 0.$$

The boundary condition

$$X(l) = 0 \implies D\sin(\beta l) = 0 \implies \beta = \frac{n\pi}{l}$$

for some integer n. Therefore, we have an infinite number of eigenvalues $\lambda_n = \beta_n^2 = (n\pi/l)^2$ with corresponding eigenfunctions

$$X_n(x) = D_n \sin\left(\frac{n\pi}{l}x\right)$$

where D_n is arbitrary.

We have proven that $\lambda_n = (n\pi/l)^2$ are eigenvalues for the eigenvalue problem (6.3). We need to check that there are no other eigenvalues.

We check if $\lambda = 0$ is an eigenvalue. If $\lambda = 0$ is an eigenvalue, our eigenvalue problem becomes

$$\begin{cases} X''(x) = 0 \\ X(0) = 0 = X(l). \end{cases} \tag{6.4}$$

The solutions of this ODE are

$$X(x) = Cx + D$$

for C, D arbitrary. The boundary condition

$$X(0) = 0 \implies D = 0.$$

The boundary condition

$$X(l) = 0 \implies C = 0.$$

Therefore, the only function X which satisfies (6.4) is $X(x) = 0$. By definition, the zero function is not an eigenfunction. Therefore, $\lambda = 0$ is not an eigenvalue of (6.3).

Next, we check if there are any negative eigenvalues. Suppose $\lambda = -\gamma^2$ is an eigenvalue of (6.3). If we have any negative eigenvalues, our eigenvalue problem becomes

$$\begin{cases} X''(x) - \gamma^2 X(x) = 0 \\ X(0) = 0 = X(l). \end{cases} \tag{6.5}$$

The solutions of this ODE are given by

$$X(x) = C\cosh(\gamma x) + D\sinh(\gamma x)$$

for C, D arbitrary. Our boundary condition

$$X(0) = 0 \implies C = 0.$$

Our boundary condition

$$X(l) = 0 \implies D = 0.$$

Therefore, the only solution of (6.5) is $X = 0$. Again, by definition, the zero function is not an eigenfunction. Therefore, there are no negative eigenvalues of (6.3).

□

Consequently, we have found an infinite sequence of eigenvalues and eigenfunctions for our eigenvalue problem (6.3). For each of these pairs $\{\lambda_n, X_n\}$, we look for a function T_n satisfying

$$-T_n''(t) = \lambda_n c^2 T_n(t). \tag{6.6}$$

Then letting $u_n(x,t) = X_n(x)T_n(t)$, we will have found a solution of the wave equation on $[0,l]$ which satisfies our boundary conditions. Of course, we have not yet looked into satisfying our initial conditions. We will consider that shortly. First, we see that for each n the solution of (6.6) is given by

$$T_n(t) = A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right).$$

Therefore, for each n,

$$\begin{aligned} u_n(x,t) &= T_n(t)X_n(x) \\ &= \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right) \end{aligned}$$

is a solution of the wave equation on the interval $[0,l]$ which satisfies $u_n(0,t) = 0 = u_n(l,t)$. More generally, using the fact that the wave equation is *linear*, we see that any finite linear combination of the functions u_n will also give us a solution of the wave equation on $[0,l]$ satisfying our Dirichlet boundary conditions. That is, any function of the form

$$u(x,t) = \sum_{n=1}^{N} \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right)$$

solves the wave equation on $[0,l]$ and satifies $u(0,t) = 0 = u(l,t)$.

Now we also want the solution to satisfy our initial conditions. Our hope is that we can choose our coefficients A_n, B_n appropriately so that $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. That is, we would like to choose A_n, B_n such that

$$\begin{aligned} u(x,0) &= \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi}{l}x\right) = \phi(x) \\ u_t(x,0) &= \sum_{n=0}^{N} B_n \frac{n\pi c}{l} \sin\left(\frac{n\pi}{l}x\right) = \psi(x). \end{aligned} \tag{6.7}$$

Our desire to express our initial data as a linear combination of trigonometric functions leads us to a couple of quesitons.

- For which functions ϕ, ψ can we find constants A_n, B_n such that we can write ϕ and ψ in the forms shown in (6.7)?
- How do we find the coefficients A_n, B_n?

In answer to our first question, for general functions ϕ, ψ we cannot express them as *finite* linear combinations of trigonometric functions. However, if we allow for *infinite* series, then for "nice" functions, we can express them as combinations of trigonometric functions. We will be more specific about what we mean by "nice" functions later. We will answer the second question, regarding finding our coefficients, shortly. First, however, we make a remark.

Remark. Stated above, we claim that we can represent "nice" functions ϕ and ψ in terms of infinite series expansions involving our eigenfunctions $\sin\left(\frac{n\pi}{l}x\right)$. By this, we mean there exist constants A_n, B_n such that

$$\begin{aligned} \phi(x) &\sim \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right) \\ \psi(x) &\sim \sum_{n=1}^{\infty} B_n \frac{n\pi c}{l} \sin\left(\frac{n\pi}{l}x\right), \end{aligned} \tag{6.8}$$

where $\sim$ means that the infinite series converge to ϕ and ψ, respectively, in some appropriate sense. We will discuss convergence issues shortly. Assuming for now that we can find sequences $\{A_n\}$, $\{B_n\}$ such that ϕ, ψ satisfy (6.8), then defining

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right),$$

we claim we will have found a solution of (6.1). We should note that if $u(x,t)$ was a *finite* sum, then it would satisfy the wave equation, as described earlier. To say that the infinite series satisfies the wave equation is a separate question. This is a technical point which we will return to later. ◇

To recap, so far we have shown that any function of the form

$$u_n(x,t) = \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right)$$

is a solution of the wave equation on $[0,l]$ which satisfies Dirichlet boundary conditions. In addition, any finite linear combination of functions of this form will also satisfy the wave equation on the interval $[0,l]$ with zero boundary conditions. We claim that

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right)$$

will also solve the wave equation on $[0, l]$. Before discussing this issue, we look for appropriate coefficients A_n, B_n which will satisfy our initial conditions.

As stated above, we would like to find coefficients A_n such that

$$\phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right). \tag{6.9}$$

Ignoring the convergence issues for a moment, if ϕ can be expressed in terms of this infinite sum, what must the coefficients A_n be? For a fixed integer m, multiply both sides of (6.9) by $\sin\left(\frac{m\pi}{l}x\right)$ and integrate from $x = 0$ to $x = l$. This leads to

$$\int_0^l \sin\left(\frac{m\pi}{l}x\right)\phi(x)\,dx = \int_0^l \sin\left(\frac{m\pi}{l}x\right)\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)\,dx. \tag{6.10}$$

Now we use the following property of the sine functions. In particular, we use the fact that

$$\boxed{\int_0^l \sin\left(\frac{m\pi}{l}x\right)\sin\left(\frac{n\pi}{l}x\right)\,dx = \begin{cases} \frac{l}{2} & m = n \\ 0 & m \neq n. \end{cases}} \tag{6.11}$$

Proof of (6.11). Recalling the trigonometric identities

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B),$$

we have

$$\sin(A)\sin(B) = \frac{1}{2}\cos(A - B) - \frac{1}{2}\cos(A + B).$$

Therefore,

$$\int_0^l \sin\left(\frac{m\pi}{l}x\right)\sin\left(\frac{n\pi}{l}x\right)\,dx = \frac{1}{2}\int_0^l \cos\left(\frac{(m-n)\pi}{l}x\right)\,dx - \frac{1}{2}\int_0^l \cos\left(\frac{(m+n)\pi}{l}x\right)\,dx.$$

First,

$$\begin{aligned}\int_0^l \cos\left(\frac{(m+n)\pi}{l}x\right)\,dx &= \frac{l}{(m+n)\pi}\sin\left(\frac{(m+n)\pi}{l}x\right)\Big|_{x=0}^{x=l} \\ &= \frac{l}{(m+n)\pi}[\sin((m+n)\pi) - \sin(0)] = 0.\end{aligned}$$

Similarly, for $m \neq n$,

$$\begin{aligned}\int_0^l \cos\left(\frac{(m-n)\pi}{l}x\right)\,dx &= \frac{l}{(m-n)\pi}\sin\left(\frac{(m-n)\pi}{l}x\right)\Big|_{x=0}^{x=l} \\ &= \frac{l}{(m-n)\pi}[\sin((m-n)\pi) - \sin(0)] = 0.\end{aligned}$$

But, for $m = n$,

$$\int_0^l \cos\left(\frac{(m-n)\pi}{l}x\right)\,dx = \int_0^l \cos(0)\,dx = \int_0^l \,dx = l.$$

Therefore, for $m \neq n$, we have

$$\int_0^l \sin\left(\frac{m\pi}{l}x\right)\sin\left(\frac{n\pi}{l}x\right)\,dx = 0,$$

while for $m = n$, we have

$$\int_0^l \sin\left(\frac{m\pi}{l}x\right)\sin\left(\frac{n\pi}{l}x\right)\,dx = \frac{l}{2},$$

as claimed. □

Now using (6.11), (6.10) becomes

$$\int_0^l \sin\left(\frac{m\pi}{l}x\right)\phi(x)\,dx = A_m\frac{l}{2}.$$

Therefore, our coefficients A_m are given by

$$A_m = \frac{2}{l}\int_0^l \sin\left(\frac{m\pi}{l}x\right)\phi(x)\,dx.$$

With these ideas in mind, for a given function ϕ, we define its **Fourier sine series** on the interval $[0, l]$ as

$$\boxed{\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)} \tag{6.12}$$

where

$$\boxed{A_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi}{l}x\right)\phi(x)\,dx.} \tag{6.13}$$

Remark. We should mention that all our estimates above have been rather formal. We have not shown yet that any function ϕ can be represented by its Fourier series. Rather, we have shown that *if* a function ϕ can be represented by the infinite series

$$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right),$$

then its coefficients A_n should be given by

$$A_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi}{l}x\right)\phi(x)\,dx.$$

The work done above has led us to the definition of the Fourier sine series associated with a given function ϕ. It still remains to determine when a Fourier sine series for a given function ϕ will actually converge to ϕ. $\diamond$

We now have enough information to put together a solution of the initial-value problem for the one-dimensional wave equation with Dirichlet boundary conditions (6.1). Combining the infinite series expansions of our initial data ϕ and ψ (6.8) with (6.13), we define coefficients

$$\boxed{\begin{aligned} A_n &\equiv \frac{2}{l}\int_0^l \sin\left(\frac{n\pi}{l}x\right)\phi(x)\,dx \\ \frac{n\pi c}{l}B_n &\equiv \frac{2}{l}\int_0^l \sin\left(\frac{n\pi}{l}x\right)\psi(x)\,dx. \end{aligned}} \tag{6.14}$$

and then our solution will be given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi}{l}ct\right) + B_n\sin\left(\frac{n\pi}{l}ct\right)\right]\sin\left(\frac{n\pi}{l}x\right).} \tag{6.15}$$

6.2 Orthogonality and Symmetric Boundary Conditions

In using the method of separation of variables in the previous section, one of the keys in finding the coefficients A_n, B_n (6.14) in the infinite series expansion (6.15) was using the fact that the eigenfunctions $\sin\left(\frac{n\pi}{l}x\right)$ satisfy

$$\int_0^l \sin\left(\frac{n\pi}{l}x\right)\sin\left(\frac{m\pi}{l}x\right)\,dx = 0$$

for $m \neq n$. In this section, we consider the wave equation on $[0,l]$ with different boundary conditions and discuss when our eigenfunctions will satisfy the same type of condition, thus allowing us to determine the coefficients in the infinite series expansion of our solution.

First, we recall some facts from linear algebra. Let $\vec{v} = [v_1,\ldots,v_n]^T$, $\vec{w} = [w_1,\ldots,w_n]^T$ be two vectors in $\mathbb{R}^n$. The dot product of $\vec{v}$ and $\vec{w}$ is defined as

$$\vec{v}\cdot\vec{w} = v_1w_1 + \ldots + v_nw_n.$$

The norm of $\vec{v}$ is given by

$$\|\vec{v}\| = \sqrt{v_1^2 + \ldots + v_n^2} = (\vec{v}\cdot\vec{v})^{1/2}.$$

We say $\vec{v}$ and $\vec{w}$ are orthogonal if their dot product is zero; that is,

$$\vec{v}\cdot\vec{w} = 0.$$

We now extend these ideas to functions. Let f, g be two real-valued functions defined on the interval $[a,b] \subset \mathbb{R}$. We define the $\mathbf{L^2}$ **inner product** of f and g on $[a,b]$ as

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx.$$

We define the $\mathbf{L^2}$ **norm** of f on the interval $[a, b]$ as

$$\|f\|_{L^2([a,b])} = \left(\int_a^b |f(x)|^2 \, dx \right)^{1/2} = \langle f, f \rangle^{1/2},$$

and define the **space of $\mathbf{L^2}$ functions on** $[a, b]$ as

$$L^2([a, b]) = \{f : \|f\|_{L^2([a,b])} < +\infty\}.$$

We say f and g are **orthogonal** on $[a, b]$ if their L^2 inner product on $[a, b]$ is zero; that is,

$$\langle f, g \rangle = \int_a^b f(x)g(x) \, dx = 0.$$

With these definitions, we note that the functions $\{\sin\left(\frac{n\pi}{l}x\right)\}$ are mutually orthogonal. We were able to use this fact to find the coefficients in the infinite series expansion of our solution u of the wave equation on $[0, l]$ with Dirichlet boundary conditions. If our eigenfunctions were not orthogonal, we would have had no way of solving for our coefficients. Suppose we consider the wave equation on $[0, l]$ with a different set of boundary conditions. Will our eigenfunctions be orthogonal?

For example, consider the initial-value problem for the wave equation on $[0, l]$ with more general boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x, 0) = \phi(x) & 0 < x < l \\ u_t(x, 0) = \psi(x) & 0 < x < l \\ u \text{ satisfies certain boundary} & \\ \quad \text{conditions at } x = 0, x = l \text{ for all } t. & \end{cases}$$

Using the method of separation of variables, we are led to the following eigenvalue problem,

$$\begin{cases} -X'' = \lambda X \qquad 0 < x < l \\ X \text{ satisfies certain boundary conditions at } x = 0, x = l. \end{cases} \tag{6.16}$$

Let $\{X_n\}$ be a sequence of eigenfunctions for this eigenvalue problem. Under what conditions will $\{X_n\}$ be a sequence of mutually orthogonal eigenfunctions? That is, when can we guarantee that

$$\langle X_n, X_m \rangle = \int_0^l X_n X_m \, dx = 0$$

for $m \neq n$?

In order to answer this question, we recall some facts from linear algebra. Let A be an $n \times n$ matrix. One condition which guarantees the orthogonality of the eigenvectors of A is symmetry of A; that is, $A = A^T$. (*Note:* A matrix A being symmetric is a *sufficient* but not necessary condition for its eigenvectors being orthogonal.) Now, we don't have a notion of symmetry for operators yet, but let's try to use the properties of symmetric matrices to extend the idea to operators. We note that if A is a symmetric matrix, then for all vectors $\vec{u}$, $\vec{v}$ in $\mathbb{R}^n$,

$$A\vec{u}\cdot\vec{v} = (A\vec{u})^T\vec{v} = \vec{u}^T A^T\vec{v} = \vec{u}^T A\vec{v} = \vec{u}\cdot A\vec{v}.$$

The converse of this statement is also true. That is, if

$$A\vec{u}\cdot\vec{v} = \vec{u}\cdot A\vec{v}$$

for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$, then A is symmetric. It is left to the reader to verify this fact. Consequently, we conclude that if

$$A\vec{u}\cdot\vec{v} = \vec{u}\cdot A\vec{v}$$

for all $\vec{u}$, $\vec{v}$ in $\mathbb{R}^n$, then the eigenvectors of A are orthogonal.

We now plan to extend the idea of symmetry of matrices to operators to find sufficient conditions for the orthogonality of eigenfunctions. Suppose we have an operator $\mathcal{L}$ such that

$$\langle \mathcal{L}(u), v\rangle = \langle u, \mathcal{L}(v)\rangle$$

for all u, v in an appropriate function space. It turns out that the eigenfunctions associated with the eigenvalue problem

$$\begin{cases} \mathcal{L}(X) = \lambda X & \vec{x}\in\Omega\subset\mathbb{R}^n \\ X \text{ satisfies certain boundary conditions on } \partial\Omega \end{cases} \tag{6.17}$$

are orthogonal. We state this more precisely in the following lemma.

Lemma 2. *Let X_n, X_m be eigenfunctions of* (6.17) *with eigenvalues λ_n, λ_m respectively such that $\lambda_n \neq \lambda_m$. If*

$$\langle \mathcal{L}(X_n), X_m\rangle = \langle X_n, \mathcal{L}(X_m)\rangle\,, \tag{6.18}$$

then X_n and X_m are orthogonal.

Proof. Combining (6.18) with the fact that X_n, X_m are eigenfunctions of (6.17) with eigenvalues λ_n, λ_m respectively, we have

$$\begin{aligned} \lambda_n \langle X_n, X_m\rangle &= \langle \mathcal{L}(X_n), X_m\rangle \\ &= \langle X_n, \mathcal{L}(X_m)\rangle \\ &= \lambda_m \langle X_n, X_m\rangle\,. \end{aligned}$$

Therefore,

$$(\lambda_n - \lambda_m)\langle X_n, X_m\rangle = 0.$$

Now using the assumption that $\lambda_n \neq \lambda_m$, we conclude that

$$\langle X_n, X_m\rangle = 0,$$

as claimed. □

Now we return to our eigenvalue problem (6.16). We would like to find sufficient conditions under which the boundary conditions lead to orthogonal eigenfunctions. We make use of the above lemma. In particular, for (6.16) our operator $\mathcal{L} = -\partial_x^2$. By the above lemma, we know that *if* X_n and X_m are eigenfunctions of (6.16) which correspond to distinct eigenvalues λ_n, λ_m and

$$\langle -X_n'', X_m \rangle = \langle X_n, -X_m'' \rangle \,, \tag{6.19}$$

then X_n and X_m are orthogonal. Therefore, if we can find a sufficient condition under which (6.19) holds, then we can prove orthogonality of eigenfunctions corresponding to distinct eigenvalues. We state such a condition in the following lemma.

Lemma 3. *If*

$$[f(x)g'(x) - f'(x)g(x)]\big|_{x=0}^{x=l} = 0 \tag{6.20}$$

for all functions f, g satisfying the boundary conditions in (6.16)*, then*

$$\langle -X_n'', X_m \rangle = \langle X_n, -X_m'' \rangle$$

for all eigenfunctions X_n, X_m of (6.16)*.*

Proof. Integrating by parts, we see that

$$\begin{aligned}
\langle X_n'', X_m \rangle &= \int_0^l X_n'' X_m \, dx \\
&= X_n' X_m\big|_{x=0}^{x=l} - \int_0^l X_n' X_m' \, dx \\
&= X_n' X_m\big|_{x=0}^{x=l} - X_n X_m'\big|_{x=0}^{x=l} + \int_0^l X_n X_m'' \, dx \\
&= [X_n' X_m - X_n X_m']\big|_{x=0}^{x=l} + \langle X_n, X_m'' \rangle \,.
\end{aligned}$$

Therefore, if (6.20) holds for all functions f, g which satisfy the boundary conditions in (6.16), then necessarily

$$[X_n' X_m - X_n X_m']\big|_{x=0}^{x=l} = 0,$$

and the lemma follows. □

Putting together the two lemmas above, we conclude that condition (6.20) is sufficient to guarantee orthogonality of eigenfunctions corresponding to distinct eigenvalues. Consequently, we define this condition as a *symmetry* condition associated with the operator $-\partial_x^2$. In particular, for the eigenvalue problem (6.16), we say the **boundary conditions are symmetric** if

$$[f(x)g'(x) - f'(x)g(x)]\big|_{x=0}^{x=l} = 0$$

for all functions f and g satisfying the given boundary conditions.

Corollary 4. *Consider the eigenvalue problem* (6.16)*. If the boundary conditions are symmetric, then eigenfunctions corresponding to distinct eigenvalues are orthogonal.*

Remark. The above corollary states that eigenfunctions corresponding to distinct eigenvalues are orthogonal. In fact, eigenfunctions associated with the same eigenvalue can be chosen to be orthogonal by using the Gram-Schmidt orthogonalization process. (See Strauss, Sec. 5.3, Exercise 10) $\diamond$

Now, let's return to solving the wave equation on an interval. Consider the initial-value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u \text{ satisfies symmetric B.C.'s} \end{cases} \tag{6.21}$$

Below we will show *formally* how to construct a solution of this problem. We will indicate how symmetric boundary conditions allow us to use separation of variables to construct a solution. Specifically, we will find a formula for the coefficients in the infinite series expansion in terms of the eigenfunctions and the initial data.

Using separation of variables, we are led to the eigenvalue problem,

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X \text{ satisfies symmetric B.C.s.} \end{cases} \tag{6.22}$$

Let $\{(\lambda_n, X_n)\}$ be all the eigenvalues and corresponding eigenfunctions for this eigenvalue problem. For each eigenvalue, we solve the following equation for T_n,

$$T_n''(t) + c^2\lambda_n T_n(t) = 0.$$

The solutions of this equation are given by

$$T_n(t) = \begin{cases} A_n \cos(\sqrt{\lambda_n}ct) + B_n \sin(\sqrt{\lambda_n}ct) & \text{if } \lambda_n > 0 \\ A_n + B_n t & \text{if } \lambda_n = 0 \\ A_n \cosh(\sqrt{-\lambda_n}ct) + B_n \sinh(\sqrt{-\lambda_n}ct) & \text{if } \lambda_n < 0, \end{cases} \tag{6.23}$$

where A_n and B_n are arbitrary. Putting each solution T_n of this equation together with the corresponding eigenfunction X_n, we arrive at a solution of the wave equation

$$u_n(x,t) = X_n(x)T_n(t)$$

which satisfies the boundary conditions. Due to the linearity of the wave equation, we know any *finite* sum of solutions u_n is also a solution. We now look to find an appropriate combination of these solutions u_n so that our initial conditions will be satisfied. That is, we would like to choose A_n, B_n appropriately for each T_n such that by defining

$$u(x,t) = \sum_n X_n(x)T_n(t) \tag{6.24}$$

with that choice A_n, B_n, our initial conditions are satisfied. In particular, we want to find coefficients A_n, B_n such that

$$u(x,0) = \sum_n X_n(x)T_n(0) = \sum_n A_n X_n(x) = \phi(x)$$

$$u_t(x,0) = \sum_n X_n(x)T_n'(0) = \sum_n C_n X_n(x) = \psi(x),$$

where

$$C_n \equiv \begin{cases} cB_n\sqrt{\lambda_n} & \lambda_n > 0 \\ B_n & \lambda_n = 0 \\ cB_n\sqrt{-\lambda_n} & \lambda_n < 0. \end{cases}$$

If we can find coefficents A_n, C_n such that we can write our initial data ϕ, ψ as

$$\begin{aligned} \phi(x) &= \sum_n A_n X_n(x) \\ \psi(x) &= \sum_n C_n X_n(x), \end{aligned} \tag{6.25}$$

we claim that we have found a solution of (6.21) given by

$$u(x,t) = \sum_n X_n(x)T_n(t)$$

where T_n is defined in (6.23) with A_n defined by (6.25) and

$$B_n \equiv \begin{cases} C_n/c\sqrt{\lambda_n} & \lambda_n > 0 \\ C_n & \lambda_n = 0 \\ C_n/c\sqrt{-\lambda_n} & \lambda_n < 0, \end{cases} \tag{6.26}$$

for C_n defined in (6.25).

Now whether we can represent our initial data in terms of our eigenfunctions is a key issue we need to consider if we hope to use this method. Luckily, it turns out that for nice functions ϕ, ψ and any symmetric boundary conditions, we *can* represent ϕ, ψ in terms of our eigenfunctions. In fact, as we will show later, for any symmetric boundary conditions, there is an infinite sequence of eigenvalues $\{\lambda_n\}$ for (6.22) such that $\lambda_n \to +\infty$ as $n \to +\infty$. Corresponding with these eigenvalues, we will have an infinite sequence of orthogonal eigenfunctions. It turns out that any L^2 function can be represented by these eigenfunctions. In addition, this infinite sequence of eigenfunctions allows us to represent our solution as

$$u(x,t) = \sum_n X_n(x)T_n(t)$$

for appropriately chosen constants A_n, B_n in the definition of T_n. We should mention that in general this solution will be an infinite series solution. It remains to prove that an *infinite* sum of solutions will still give us a solution of (6.21). We will return to these issues later.

First, however, let us try to find formulas for our coefficients A_n, C_n in (6.25). Assuming we can represent our initial data in terms of our eigenfunctions as in (6.25), we can use Corollary 4 to determine what our coefficients should be. Specifically,

$$\begin{aligned} \phi(x) &= \sum_{n=1}^{\infty} A_n X_n(x) \\ \implies \left\langle X_m, \sum_{n=1}^{\infty} A_n X_n \right\rangle &= \langle X_m, \phi \rangle . \end{aligned}$$

Now by Corollary (6.25), we know that symmetric boundary conditions imply eigenfunctions corresponding to distinct eigenvalues are orthogonal, and, consequently,

$$\langle X_m, A_m X_m \rangle = \langle X_m, \phi \rangle .$$

Therefore, the coefficients A_m for ϕ in the infinite series expansion

$$\phi(x) = \sum_n A_n X_n(x)$$

must be given by

$$\boxed{A_m = \frac{\langle X_m, \phi \rangle}{\langle X_m, X_m \rangle}.} \tag{6.27}$$

Similarly, the coefficients C_m in the infinite series expansion

$$\psi(x) = \sum_n C_n X_n(x)$$

must be given by

$$\boxed{C_m = \frac{\langle X_m, \psi \rangle}{\langle X_m, X_m \rangle}.} \tag{6.28}$$

Therefore, we claim that our solution of (6.21) is given by

$$u(x,t) = \sum_n X_n(x) T_n(t)$$

where T_n is defined in (6.23) with A_n defined by (6.27) and B_n defined by (6.26) for C_n defined by (6.28). Let's look at an example.

Example 5. Consider the initial-value problem for the wave equation on an interval with *Neumann boundary conditions*,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = \phi(x), & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u_x(0,t) = 0 = u_x(l,t). \end{cases} \tag{6.29}$$

Using the separation of variables technique, we are led to the eigenvalue problem

$$\begin{cases} -X'' = \lambda X \\ X'(0) = 0 = X'(l). \end{cases} \tag{6.30}$$

We note that the Neumann boundary conditions are symmetric, because for all functions f, g such that $f'(0) = 0 = g'(0)$ and $f'(l) = 0 = g'(l)$, we have

$$[f(x)g'(x) - f'(x)g(x)]\big|_{x=0}^{x=l} = 0.$$

Therefore, we will be able to use the orthogonality of the eigenfunctions to determine the coefficients of our solution in the infinite series expansion.

First, we look for all our eigenvalues and eigenfunctions of (6.30). We start by looking for positive eigenvalues $\lambda = \beta^2 > 0$. In this case, our eigenvalue problem becomes

$$\begin{cases} X'' + \beta^2 X = 0 \\ X'(0) = 0 = X'(l). \end{cases}$$

The solutions of our ODE are given by

$$X(x) = C\cos(\beta x) + D\sin(\beta x).$$

The boundary condition

$$X'(0) = 0 \implies D = 0.$$

The boundary condition

$$X'(l) = 0 \implies -C\beta\sin(\beta l) = 0 \implies \beta = \frac{n\pi}{l}.$$

Therefore, we have an infinite sequence of positive eigenvalues given by $\lambda_n = \beta_n^2 = (n\pi/l)^2$ with corresponding eigenfunctions $X_n(x) = C_n \cos\left(\frac{n\pi}{l}x\right)$.

Next, we check if $\lambda = 0$ is an eigenvalue. If $\lambda = 0$, our eigenvalue problem (6.30) becomes

$$\begin{cases} X'' = 0 \\ X'(0) = 0 = X'(l). \end{cases}$$

The solutions of this ODE are given by

$$X(x) = Cx + D,$$

where C, D are arbitrary. The boundary condition

$$X'(0) = 0 \implies C = 0.$$

The boundary condition

$$X'(l) = 0$$

is automatically satisfied as long as $C = 0$ (i.e. - the first boundary condition is satisfied). Therefore, $X(x) = D$ is an eigenfunction with eigenvalue $\lambda = 0$.

Last, we look for negative eigenvalues. That is, we look for an eigenvalue $\lambda = -\gamma^2$. In this case, our eigenvalue problem (6.30) becomes

$$\begin{cases} X'' - \gamma^2 X = 0 \\ X'(0) = 0 = X'(l). \end{cases}$$

The solutions of the ODE are given by

$$X(x) = C\cosh(\gamma x) + D\sinh(\gamma x).$$

The boundary condition

$$X'(0) = 0 \implies D = 0.$$

The boundary condition

$$X'(l) = 0 \implies C\gamma \sinh(\gamma l) = 0 \implies C = 0.$$

Therefore, the only function X which satisfies our eigenvalue problem for $\lambda = -\gamma^2 < 0$ is the zero function, which by definition is not an eigenfunction. Therefore, we have no negative eigenvalues.

Consequently, the eigenvalues and eigenfunctions for (6.30) are given by

$$\boxed{\begin{array}{lll} \lambda_n = \left(\frac{n\pi}{l}\right)^2, & X_n(x) = \cos\left(\frac{n\pi}{l}x\right), & n = 1, 2, \ldots \\ \lambda_0 = 0, & X_0(x) = 1. & \end{array}}$$

For each $n = 0, 1, 2, \ldots$, we need to look for a solution of our equation for T_n. In particular, we need to solve

$$T_n''(t) + c^2 \lambda_n T_n(t) = 0.$$

As described earlier, the solutions of this equation are given by

$$\boxed{\begin{array}{l} T_n(t) = A_n \cos\left(\dfrac{n\pi}{l}ct\right) + B_n \sin\left(\dfrac{n\pi}{l}ct\right) \quad n = 1, 2, \ldots \\ T_0(t) = A_0 + B_0 t. \end{array}}$$

for A_n, B_n arbitrary.

Putting these functions X_n, T_n together, we propose that our solution is given by

$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \cos\left(\frac{n\pi}{l}x\right)$$

for appropriately chosen constants A_n, B_n. Using our initial conditions, we want to choose A_n, B_n such that

$$\begin{aligned} u(x,0) &= A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l}x\right) = \phi(x) \\ u_t(x,0) &= B_0 + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n \cos\left(\frac{n\pi}{l}x\right) = \psi(x). \end{aligned} \tag{6.31}$$

That is, we want constants A_n, B_n such that

$$\begin{aligned} \phi(x) &= \sum_{n=0}^{\infty} A_n X_n(x) \\ \psi(x) &= B_0 X_0(x) + \sum_{n=1}^{\infty} \frac{n\pi c}{l} B_n X_n(x). \end{aligned}$$

Note: For ψ, we look for constants C_n such that

$$\psi(x) = \sum_{n=0}^{\infty} C_n X_n(x)$$

and then define B_n such that

$$B_n = \begin{cases} C_0 & n = 0 \\ \frac{l}{n\pi c} C_n & n = 1, 2, \ldots \end{cases}$$

Using the fact that the Neumann boundary conditions are symmetric, we know that eigenfunctions corresponding to distinct eigenvalues must be orthogonal. Consequently, we can use the formulas (6.27), (6.28) derived earlier for A_n, C_n. In particular,

$$A_n \equiv \frac{\langle X_n, \phi \rangle}{\langle X_n, X_n \rangle}$$
$$C_n \equiv \frac{\langle X_n, \psi \rangle}{\langle X_n, X_n \rangle}$$

Now

$$\langle X_0, \phi \rangle = \int_0^l X_0 \phi \, dx = \int_0^l \phi(x) \, dx$$
$$\langle X_n, \phi \rangle = \int_0^l X_n \phi \, dx = \int_0^l \cos\left(\frac{n\pi}{l} x\right) \phi(x) \, dx \quad n = 1, 2, \ldots$$

and

$$\langle X_0, X_0 \rangle = \int_0^l dx = l$$
$$\langle X_n, X_n \rangle = \int_0^l \cos^2\left(\frac{n\pi}{l} x\right) dx = \frac{l}{2} \qquad n = 1, 2, \ldots$$

Therefore, our solution of (6.29) is given by

$$\boxed{u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l} ct\right) + B_n \sin\left(\frac{n\pi}{l} ct\right) \right] \cos\left(\frac{n\pi}{l} x\right)}$$

where

$$\boxed{\begin{aligned} A_0 &= \frac{\langle X_0, \phi \rangle}{\langle X_0, X_0 \rangle} = \frac{1}{l} \int_0^l \phi(x) \, dx \\ A_n &= \frac{\langle X_n, \phi \rangle}{\langle X_n, X_n \rangle} = \frac{2}{l} \int_0^l \cos\left(\frac{n\pi}{l} x\right) \phi(x) \, dx \quad n = 1, 2, \ldots \end{aligned}}$$

and

$$\boxed{\begin{aligned} B_0 = C_0 &= \frac{\langle X_0, \psi \rangle}{\langle X_0, X_0 \rangle} = \frac{1}{l} \int_0^l \psi(x) \, dx \\ \frac{n\pi c}{l} B_n = C_n &= \frac{\langle X_0, \psi \rangle}{\langle X_n, X_n \rangle} = \frac{2}{l} \int_0^l \cos\left(\frac{n\pi}{l} x\right) \psi(x) \, dx \quad n = 1, 2, \ldots \end{aligned}}$$

Remarks.

- Using the fact that our boundary conditions are symmetric, we know from Corollary 4 that eigenfunctions corresponding to distinct eigenvalues are orthogonal. As a result, we gain *for free* the orthogonality property of cosine functions:

$$\int_0^l \cos\left(\frac{m\pi}{l}x\right)\cos\left(\frac{n\pi}{l}x\right) = 0 \qquad \text{for } m \neq n.$$

- The series

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l}x\right)$$

  where

$$\begin{aligned} A_0 &\equiv \frac{1}{l}\int_0^l \phi(x)\,dx \\ A_n &\equiv \frac{2}{l}\int_0^l \cos\left(\frac{n\pi}{l}x\right)\phi(x)\,dx \quad n = 1, 2, \ldots \end{aligned}$$

  is known as the **Fourier cosine series** of ϕ on the interval $[0, l]$.

◇

We close this section by giving some examples of symmetric boundary conditions,

Example 6. The following boundary conditions are symmetric for the eigenvalue problem (6.22).

- Dirichlet: $X(0) = 0 = X(l)$
- Neumann: $X'(0) = 0 = X'(l)$
- Periodic: $X(0) = X(l)$, $X'(0) = X'(l)$
- Robin: $X'(0) = aX(0)$, $X'(l) = bX(l)$

◇

6.3 Fourier Series

In the previous section we showed that solving an initial-value problem for the wave equation on an interval can essentially be reduced to studying an associated eigenvalue problem. That is, using separation of variables, the IVP

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u \text{ satisfies certain B.C.s} & \end{cases} \tag{6.32}$$

can be reduced to studying the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X \text{ satisfies certain B.C.s} \end{cases}$$

One of the key ingredients in using the solutions of the eigenvalue problem to solve the initial-value problem (6.32) was being able to represent our initial data in terms of our eigenfunctions. That is, to be able to find coefficients A_n, C_n such that

$$\phi(x) = \sum_n A_n X_n(x)$$
$$\psi(x) = \sum_n C_n X_n(x).$$

In this section, we discuss this issue in detail, showing when such a representation makes sense.

First, we start with some definitions. For a function ϕ defined on $[0, l]$ we define its **Fourier sine series** as

$$\phi(x) \sim \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)$$

where

$$A_n \equiv \frac{\langle \sin(n\pi x/l), \phi \rangle}{\langle \sin(n\pi x/l), \sin(n\pi x/l) \rangle}.$$

Note: As we have not discussed any convergence issues yet, the notation "∼" should just be thought of as meaning ϕ is associated with the Fourier series shown. We saw this series earlier in the case of Dirichlet boundary conditions. In the case of Neumann boundary conditions, we were led to the **Fourier cosine series** of ϕ on $[0, l]$,

$$\phi(x) \sim \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi}{l}x\right)$$

where

$$A_n \equiv \frac{\langle \cos(n\pi x/l), \phi \rangle}{\langle \cos(n\pi x/l), \cos(n\pi x/l) \rangle}$$

The other classical Fourier series is known as the **full Fourier series**. The full Fourier series for ϕ defined on $[-l, l]$ is given by

$$\phi(x) \sim A_0 + \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l}x\right) + B_n \sin\left(\frac{n\pi}{l}x\right) \right],$$

where

$$\begin{aligned} A_n &\equiv \frac{\langle \cos(n\pi x/l), \phi \rangle}{\langle \cos(n\pi x/l), \cos(n\pi x/l) \rangle} \quad n = 0, 1, 2, \ldots \\ B_n &\equiv \frac{\langle \sin(n\pi x/l), \phi \rangle}{\langle \sin(n\pi x/l), \sin(n\pi x/l) \rangle} \quad n = 1, 2, \ldots . \end{aligned} \tag{6.33}$$

Note: The inner products for the full Fourier series are taken on the interval $[-l, l]$. The full Fourier series arises in the case of periodic boundary conditions.

Relationship between Fourier sine, Fourier cosine and full Fourier series.

Let ϕ be an odd function defined on $[-l, l]$. Then its full Fourier series is an odd function because the coefficients A_n defined in (6.33) are zero. Therefore, for ϕ odd, its full Fourier series is given by

$$\phi(x) \sim \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{l}x\right),$$

where B_n is defined in (6.33). In particular, for an odd function ϕ defined on $[-l, l]$, the full Fourier series of ϕ is the Fourier sine series of the restriction of ϕ to $[0, l]$.

Equivalently, for a function ϕ defined on $[0, l]$ the Fourier sine series of ϕ is the full Fourier series of the odd extension of ϕ to $[-l, l]$.

Similarly, for an even function defined on $[-l, l]$, the full Fourier series is an even function because the coefficients B_n defined in (6.33) are zero. Therefore, for ϕ even, its full Fourier series is

$$\phi(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi}{l}x\right),$$

where A_n is defined in (6.33). For an even function ϕ defined on $[-l, l]$, the full Fourier series of ϕ is the Fourier cosine series of the restriction of ϕ to $[0, l]$.

Equivalently, for a function ϕ defined on $[0, l]$, the Fourier cosine series of ϕ is the full Fourier series of the even extension of ϕ to $[-l, l]$.

We will use these relationships below when we discuss convergence issues.

General Fourier Series.

More generally than the classical Fourier series discusses above, we introduce the notion of a *generalized Fourier series*. Let $\{X_n\}$ be a sequence of mutually orthogonal eigenfunctions in $L^2([a, b])$. Let ϕ be any function defined on $[a, b]$. Let

$$A_n \equiv \frac{\langle X_n, \phi \rangle}{\langle X_n, X_n \rangle}.$$

Then the A_n are the **generalized Fourier coefficients** for

$$\sum_{n=1}^{\infty} A_n X_n(x),$$

a **generalized Fourier series** of ϕ on $[a, b]$.

We now state and prove some results regarding the convergence of Fourier series. First, we define some different types of convergence.

Types of Convergence.

There are several types of convergence one can consider. We define three different types. Let $\{s_n\}$ be a sequence of functions defined on the interval $[a, b]$.

- We say s_n converges to f **pointwise** on (a, b) if

$$|s_n(x) - f(x)| \to 0 \qquad \text{as } n \to +\infty$$

for each $x \in (a, b)$.

- We say s_n converges to f **uniformly** on $[a, b]$ if

$$\max_{a \leq x \leq b} |s_n(x) - f(x)| \to 0 \qquad \text{as } n \to +\infty.$$

- We say s_n converges to f in the $\mathbf{L^2}$**-sense** on (a, b) if

$$\|s_n - f\|_{L^2} \to 0 \qquad \text{as } n \to +\infty.$$

Some Convergence Results for Fourier Series.

We will now state some convergence results for Fourier series with respect to the different types of convergence. First, we need to give some definitions. We say

$$f(x^+) = \lim_{x \to x^+} f(x)$$
$$f(x^-) = \lim_{x \to x^-} f(x).$$

Theorem 7. (Pointwise Convergence of Classical Fourier Series) *(Ref: Strauss, Sec. 5.4) Let f, f' be piecewise continuous functions on $[-l, l]$. Then the full Fourier series converges to*

$$\frac{1}{2}[f_{\text{ext}}(x^+) + f_{\text{ext}}(x^-)]$$

at every point $x \in \mathbb{R}$, where f_{ext} is the $2l$-periodic extension of f.

Remark. Due to the relationship between the Fourier sine, Fourier cosine and full Fourier series, this theorem implies the convergence of the Fourier sine and Fourier cosine series as well. For a function ϕ defined on $[0, l]$, the Fourier sine series of ϕ converges if and only if the full Fourier series of the odd extension of ϕ to $[-l, l]$ converges.

The following two theorems allow for generalized Fourier series. Let f be any function defined on $[a, b]$. Let $\{X_n\}$ be a sequence of mutually orthogonal eigenfunctions for the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & a < x < b \\ X \ \text{ satisfies symmetric B.C.s} \end{cases}$$

Let $\sum_n A_n X_n(x)$ be the associated generalized Fourier series of f.

Theorem 8. (Uniform Convergence) *(Ref: Strauss, Sec. 5.4) The Fourier series $\sum_n A_n X_n(x)$ converges to f uniformly on $[a, b]$ provided that*

(1) f, f', and f'' exist and are continuous for $x \in [a, b]$.

(2) f satisfies the given boundary conditions.

Remark. In the case of any of the classical Fourier series, the assumption that f'' exists may be dropped.

Theorem 9. (L^2 convergence) *(Ref: Strauss, Sec. 5.4) The Fourier series $\sum_n A_n X_n(x)$ converges to f in the L^2 sense in (a, b) as long as $f \in L^2([a, b])$.*

Proofs of Convergence Results.

In this section, we prove Theorem 7. In order to do so, we need to prove some preliminary results.

Theorem 10. *(Ref: Strauss, Sec. 5.4) Let $\{X_n\}$ be a sequence of mutually orthogonal functions in L^2. Let $f \in L^2$. Fix a positive integer N. Then the expression*

$$E_N = \left\| f - \sum_{n=1}^{N} a_n X_n \right\|_{L^2}^2$$

is minimized by choosing

$$a_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle}.$$

Remark. This means that in the series $\sum_n a_n X_n$, the coefficients a_n which provide the best least-squares approximation to f are the Fourier coefficients.

Proof. Using the properties of inner product and the orthogonality of the functions X_n, we have

$$\begin{aligned}
E_N &= \left\| f - \sum_{n=1}^{N} a_n X_n \right\|_{L^2}^2 \\
&= \left\langle f - \sum_{n=1}^{N} a_n X_n, f - \sum_{n=1}^{N} a_n X_n \right\rangle \\
&= \langle f, f \rangle - 2 \left\langle f, \sum_{n=1}^{N} a_n X_n \right\rangle + \left\langle \sum_{n=1}^{N} a_n X_n, \sum_{n=1}^{N} a_n X_n \right\rangle \\
&= \|f\|^2 - 2 \left\langle f, \sum_{n=1}^{N} a_n X_n \right\rangle + \sum_{m=1}^{N} \left\langle a_m X_m, \sum_{n=1}^{N} a_n X_n \right\rangle \\
&= \|f\|^2 - 2 \left\langle f, \sum_{n=1}^{N} a_n X_n \right\rangle + \sum_{m=1}^{N} \langle a_m X_m, a_m X_m \rangle \\
&= \|f\|^2 - 2 \sum_{m=1}^{N} \langle f, X_m \rangle a_m + \sum_{m=1}^{N} \langle X_m, X_m \rangle a_m^2.
\end{aligned}$$

Now completing the square in a_m, we have

$$\sum_{m=1}^{N} \langle X_m, X_m \rangle \left[a_m^2 - 2 \frac{\langle f, X_m \rangle}{\langle X_m, X_m \rangle} a_m \right] = \sum_{m=1}^{N} \langle X_m, X_m \rangle \left[\left(a_m - \frac{\langle f, X_m \rangle}{\langle X_m, X_m \rangle} \right)^2 - \frac{\langle f, X_m \rangle^2}{\langle X_m, X_m \rangle^2} \right].$$

Substituting this expression into the expression for E_N above, we see that

$$E_N = \|f\|^2 + \sum_{m=1}^{N} \langle X_m, X_m \rangle \left[\left(a_m - \frac{\langle f, X_m \rangle}{\langle X_m, X_m \rangle} \right)^2 - \frac{\langle f, X_m \rangle^2}{\langle X_m, X_m \rangle^2} \right]. \tag{6.34}$$

Therefore, the choice of a_m that will minimize E_N is clearly

$$a_m = \frac{\langle f, X_m \rangle}{\langle X_m, X_m \rangle},$$

as claimed.

□

Corollary 11. (**Bessel's Inequality**) *Let $\{X_n\}$ be a sequence of mutually orthogonal functions in L^2. Let $f \in L^2$ and let A_n be the Fourier coefficients associated with X_n; that is,*

$$A_n = \frac{\langle f, X_n \rangle}{\langle X_n, X_n \rangle}.$$

Then

$$\boxed{\sum_{n=1}^{\infty} A_n^2 \int_a^b |X_n(x)|^2 \, dx \leq \int_a^b |f(x)|^2 \, dx.} \tag{6.35}$$

Proof. Let $a_m = A_m$ in (6.34). Combining this choice of coefficients with the fact that E_N is nonnegative, we have

$$0 \leq E_N = \|f\|^2 - \sum_{m=1}^{N} \frac{\langle f, X_m \rangle^2}{\langle X_m, X_m \rangle^2} \langle X_m, X_m \rangle = \|f\|^2 - \sum_{m=1}^{N} A_m^2 \langle X_m, X_m \rangle. \tag{6.36}$$

Therefore, we conclude that

$$\sum_{m=1}^{N} A_m^2 \langle X_m, X_m \rangle \leq \|f\|^2.$$

Taking the limit as $N \to +\infty$, the corollary follows. □

Remark. Bessel's inequality will play a key role in the proof of Theorem 7 on pointwise convergence.

Theorem 12. (Parseval's Equality) *Let $\{X_n\}$ be a sequence of mutually orthogonal functions in L^2. Let $f \in L^2$. The Fourier series $\sum_n A_n X_n(x)$ converges to f in L^2 if and only if*

$$\boxed{\sum_{n=1}^{\infty} A_n^2 \|X_n\|_{L^2}^2 = \|f\|_{L^2}^2.} \tag{6.37}$$

Proof. By definition, the Fourier series $\sum_n A_n X_n$ for f converges in L^2 if and only if

$$\left\| f - \sum_{n=1}^{N} A_n X_n \right\|_{L^2}^2 \to 0 \qquad \text{as } N \to +\infty.$$

But, this means $E_N \to 0$ as $N \to +\infty$. Combining this with (6.36), the theorem follows. $\square$

We now have the necessary ingredients to prove Theorem 7 on pointwise convergence, but first we introduce some notation. Without loss of generality, we may assume $l = \pi$. Then the full Fourier series on $[-\pi, \pi]$ for the function f is given by

$$f(x) \sim A_0 + \sum_{n=1}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]$$

where the coefficients are given by

$$\begin{aligned}
A_0 &= \frac{\langle 1, f\rangle}{\langle 1, 1\rangle} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\
A_n &= \frac{\langle \cos(nx), f\rangle}{\langle \cos(nx), \cos(nx)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(nx) f(x)\,dx \qquad n = 1, 2, \ldots \\
B_n &= \frac{\langle \sin(nx), f\rangle}{\langle \sin(nx), \sin(nx)\rangle} = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin(nx) f(x)\,dx \qquad n = 1, 2, \ldots.
\end{aligned}$$

Let $S_N(x)$ be the N^{th} partial sum of the full Fourier series. That is,

$$S_N(x) = A_0 + \sum_{n=1}^{N} [A_n \cos(nx) + B_n \sin(nx)].$$

We need to show that S_N converges pointwise to

$$\frac{1}{2}[f(x^+) + f(x^-)] \qquad -\pi < x < \pi.$$

Substituting the formulas for the coefficients into S_N, we see that

$$\begin{aligned}
S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[1 + 2\sum_{n=1}^{N} (\cos(nx)\cos(ny) + \sin(nx)\sin(ny))\right] f(y)\,dy \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[1 + 2\sum_{n=1}^{N} \cos(n(y - x))\right] f(y)\,dy.
\end{aligned}$$

The term

$$K_N(\theta) = 1 + 2\sum_{n=1}^{N} \cos(n\theta)$$

is called the **Dirichlet kernel**. With this definition, we write

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y - x) f(y)\,dy.$$

We list two properties of the Dirichlet kernel which will be useful in proving Theorem 7.

- $\int_{-\pi}^{0} K_n(\theta)\,d\theta = \int_{0}^{\pi} K_n(\theta)\,d\theta = \pi$ for all positive integers N.
- $K_N(\theta) = \dfrac{\sin\left(\left[N+\frac{1}{2}\right]\theta\right)}{\sin\left(\frac{1}{2}\theta\right)}$.

The first property is an easy calculation. The second property can be proven as follows.

$$
\begin{aligned}
K_N(\theta) &= 1 + 2\sum_{n=1}^{N} \cos(n\theta) \\
&= 1 + \sum_{n=1}^{N} (e^{in\theta} + e^{-in\theta}) \\
&= \sum_{n=-N}^{N} e^{in\theta} \\
&= \frac{e^{-iN\theta} - e^{i(N+1)\theta}}{1 - e^{i\theta}} \\
&= \frac{e^{-i(N+\frac{1}{2})\theta} - e^{i(N+\frac{1}{2})\theta}}{e^{-i\frac{1}{2}\theta} - e^{i\frac{1}{2}\theta}} \\
&= \frac{\sin([N+\frac{1}{2}]\theta)}{\sin\left(\frac{1}{2}\theta\right)}.
\end{aligned}
$$

Proof of Theorem 7. We need to show that for all $x \in [-\pi, \pi]$ that $S_N(x)$ converges to $\frac{1}{2}[f(x^+) - f(x^-)]$. Fix a point $x \in [-\pi, \pi]$.

Taking the 2π-periodic extension of f, we have

$$
\begin{aligned}
S_N(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(y-x) f(y)\,dy \\
&= \frac{1}{2\pi}\int_{-\pi-x}^{\pi-x} K_N(\theta) f(\theta + x)\,d\theta \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(\theta) f(\theta + x)\,d\theta.
\end{aligned}
$$

Now using the first property of $K_N(\theta)$ given above, we have

$$
\begin{aligned}
f(x^+) &= \frac{1}{\pi}\int_{0}^{\pi} K_N(\theta) f(x^+)\,d\theta \\
f(x^-) &= \frac{1}{\pi}\int_{-\pi}^{0} K_N(\theta) f(x^-)\,d\theta.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
S_N(x) - \frac{1}{2}[f(x^+) + f(x^-)] = \frac{1}{2\pi}\int_{0}^{\pi} K_N(\theta)[f(\theta + x) - f(x^+)]\,d\theta \\
+ \frac{1}{2\pi}\int_{-\pi}^{0} K_N(\theta)[f(\theta + x) - f(x^-)]\,d\theta.
\end{aligned}
$$

Now we claim that as $N \to +\infty$ both of the terms on the right-hand side tend to zero, giving us the desired result. We prove this for the first term. The second term can be handled similarly.

Defining

$$g(\theta) = \frac{f(x+\theta) - f(x^+)}{\sin(\frac{1}{2}\theta)},$$

and using the second property of $K_N(\theta)$ above, we have

$$\frac{1}{2\pi} \int_0^{\pi} K_N(\theta)[f(\theta + x) - f(x^+)]\, d\theta = \frac{1}{2\pi} \int_0^{\pi} g(\theta) \sin\left(\left[N + \frac{1}{2}\right]\theta\right)\, d\theta.$$

Defining

$$\phi_N(\theta) = \sin\left(\left[N + \frac{1}{2}\right]\theta\right),$$

it remains to show that $\langle g, \phi_N \rangle \to 0$ as $N \to +\infty$. In order to prove this, we will make use of Bessel's inequality (6.35). We do so as follows.

First, we claim that $\{\phi_N\}$ is a mutually orthogonal sequence of functions in $L^2([0,\pi])$. It can easily be verified that

$$||\phi_N||^2_{L^2([0,\pi])} = \frac{\pi}{2},$$

and, therefore, $\phi_N \in L^2([0,\pi])$. The orthogonality can be proven in the same manner that we proved the orthogonality of $\{\sin(N\pi x/l)\}$ on $[0,l]$.

Second, we claim that $g \in L^2([0,\pi])$. By assumption f is piecewise continuous. Therefore, the only potential reason why g would not be in L^2 is the singularity in g at $\theta = 0$. Therefore, we need to look at $\lim_{\theta \to 0^+} g(\theta)$. We have

$$\begin{aligned}
\lim_{\theta \to 0^+} g(\theta) &= \lim_{\theta \to 0^+} \frac{f(x+\theta) - f(x^+)}{\sin(\frac{1}{2}\theta)} \\
&= \lim_{\theta \to 0^+} \frac{f(x+\theta) - f(x^+)}{\theta} \cdot \lim_{\theta \to 0^+} \frac{\theta}{\sin(\frac{1}{2}\theta)} \\
&= 2f'(x^+).
\end{aligned}$$

By assumption, f' is piecewise continuous. Therefore, g is piecewise continuous and consequently $g \in L^2([0,\pi])$.

Using the fact that $g \in L^2([0,\pi])$ and $\{\phi_N\}$ is a mutually orthogonal sequence of functions in $L^2([0,\pi])$, we can make use of Bessel's inequality. In particular, we have

$$\sum_{N=1}^{\infty} \frac{\langle g, \phi_N \rangle^2}{\langle \phi_N, \phi_N \rangle} \leq \|g\|^2_{L^2},$$

which implies

$$\frac{2}{\pi} \sum_{N=1}^{\infty} \langle g, \phi_N \rangle^2 \leq \|g\|^2_{L^2}.$$

Now using the fact that $g \in L^2$, we conclude that $\sum_{N=1}^{\infty} \langle g, \phi_N \rangle^2$ is a convergent sequence, and, therefore, $\langle g, \phi_N \rangle \to 0$ as $N \to +\infty$. This proves the theorem. □

For proofs of the other convergence theorems, see Strauss, Section 5.5. We now show that the infinite series we claimed were solutions of the wave equation *are* actually solutions. We demonstrate this using the following example.

Example 13. We return to considering the initial-value problem for the wave equation on $[0, l]$ with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x, 0) = \phi(x) & 0 < x < l \\ u_t(x, 0) = \psi(x) & 0 < x < l \\ u(0, t) = 0 = u(l, t). \end{cases} \tag{6.38}$$

As shown earlier, any function of the form

$$u_n(x, t) = \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right)$$

satisfies the wave equation and the Dirichlet boundary conditions. Now let

$$u(x, t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right), \tag{6.39}$$

where

$$\begin{aligned} A_n &\equiv \frac{2}{l} \int_0^l \sin\left(\frac{n\pi}{l}x\right) \phi(x)\, dx \\ \frac{n\pi c}{l} B_n &\equiv \frac{2}{l} \int_0^l \sin\left(\frac{n\pi}{l}x\right) \psi(x)\, dx. \end{aligned}$$

We claim that this infinite series u is a solution of (6.38). We see that u satisfies the boundary conditions and the initial conditions (assuming "nice" initial data to allow for our convergence theorems). Therefore, it remains only to verify that this infinite series satisfies our PDE. We prove so as follows.

Let ϕ_{ext}, ψ_{ext} be the odd periodic extensions of ϕ and ψ, respectively. Consider the initial-value problem for the wave equation on the whole real line,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & -\infty < x < \infty \\ u(x, 0) = \phi_{\text{ext}}(x) \\ u_t(x, 0) = \psi_{\text{ext}}(x). \end{cases} \tag{6.40}$$

By d'Alembert's formula, the solution of (6.40) is given by

$$v(x, t) = \frac{1}{2}[\phi_{\text{ext}}(x + ct) + \phi_{\text{ext}}(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi_{\text{ext}}(y)\, dy.$$

Let $\widetilde{u}(x, t) = v(x, t)$ restricted to $0 \leq x \leq l$. First, we claim that $\widetilde{u}(x, t)$ is a solution of (6.38). Second, we claim that $\widetilde{u}(x, t)$ is given by the series expansion in (6.39), thus proving that the infinite series (6.39) is a solution of (6.38). It is easy to verify that $\widetilde{u}$ is a solution of (6.38). Therefore, it remains only to show that $\widetilde{u}(x,t)$ is given by the series expansion in (6.39). We do so as follows.

Now for ϕ defined on $[0,l]$, the Fourier sine series of ϕ is given by

$$\phi(y) \sim \sum_{n=1}^{\infty} \widetilde{A_n} \sin\left(\frac{n\pi}{l} y\right)$$

where

$$\widetilde{A_n} = \frac{\left\langle \sin\left(\frac{n\pi}{l}x\right), \phi \right\rangle}{\left\langle \sin\left(\frac{n\pi}{l}x\right), \sin\left(\frac{n\pi}{l}x\right) \right\rangle} = \frac{2}{l} \int_0^l \sin\left(\frac{n\pi}{l}x\right) \phi(x)\, dx.$$

Now for a nice function ϕ, the Fourier sine series converges to ϕ on $[0,l]$. In addition, using the fact that ϕ_{ext} is the odd periodic extension of ϕ, we know that the Fourier sine series above converges to ϕ_{ext} on all of $\mathbb{R}$. We can do a similar analysis for ψ. That is, the Fourier sine series of ψ is given by

$$\psi(y) \sim \sum_{n=1}^{\infty} \widetilde{B_n} \sin\left(\frac{n\pi}{l} y\right)$$

where

$$\widetilde{B_n} = \frac{2}{l} \int_0^l \sin\left(\frac{n\pi}{l}x\right) \psi(x)\, dx.$$

Now plugging these series representations into the formula for $\widetilde{u}(x,t)$, we have

$$\begin{aligned}
\widetilde{u}(x,t) &= \frac{1}{2}\left[\sum_{n=1}^{\infty} \widetilde{A_n} \sin\left(\frac{n\pi}{l}(x+ct)\right) + \widetilde{A_n} \sin\left(\frac{n\pi}{l}(x-ct)\right)\right] \\
&\quad + \frac{1}{2c} \int_{x-ct}^{x+ct} \sum_{n=1}^{\infty} \widetilde{B_n} \sin\left(\frac{n\pi}{l} y\right)\, dy \\
&= \frac{1}{2}\left[\sum_{n=1}^{\infty} \widetilde{A_n} 2 \sin\left(\frac{n\pi}{l}x\right) \cos\left(\frac{n\pi}{l}ct\right)\right] - \frac{l}{2n\pi c} \sum_{n=1}^{\infty} \widetilde{B_n} \cos\left(\frac{n\pi}{l} y\right)\Big|_{y=x-ct}^{y=x+ct} \\
&= \sum_{n=1}^{\infty} \widetilde{A_n} \sin\left(\frac{n\pi}{l}x\right) \cos\left(\frac{n\pi}{l}ct\right) + \frac{l}{n\pi c} \sum_{n=1}^{\infty} \widetilde{B_n} \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{n\pi}{l}ct\right).
\end{aligned}$$

But, this means $\widetilde{u}(x,t)$ can be written as

$$\widetilde{u}(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{n\pi}{l}ct\right) + B_n \sin\left(\frac{n\pi}{l}ct\right)\right] \sin\left(\frac{n\pi}{l}x\right)$$

where

$$\begin{aligned}
A_n &= \widetilde{A_n} = \frac{2}{l} \int_0^l \sin\left(\frac{n\pi}{l}x\right) \phi(x)\, dx \\
B_n &= \frac{l}{n\pi c} \widetilde{B_n} = \frac{2}{n\pi c} \int_0^l \sin\left(\frac{n\pi}{l}x\right) \psi(x)\, dx.
\end{aligned}$$

Therefore, we have shown that the infinite series (6.39) is actually a solution of (6.38).

◇

6.4 The Inhomogeneous Problem on an Interval

We now consider the inhomogeneous wave equation on an interval with symmetric boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u \text{ satisfies symmetric BCs.} \end{cases} \tag{6.41}$$

We will solve this using Duhamel's principle. Recall from our earlier discussion that the solution of an IVP for an inhomogeneous evolution equation of the form

$$\begin{cases} U_t + AU = F & \vec{x} \in \mathbb{R}^n \\ U(\vec{x},0) = \Phi(\vec{x}) \end{cases}$$

is given by

$$U(\vec{x},t) = S(t)\Phi(\vec{x}) + \int_0^t S(t-s)F(\vec{x},s)\,ds$$

where $S(t)$ is the solution operator for the homogeneous equation

$$\begin{cases} U_t + AU = 0 \\ U(\vec{x},0) = \Phi(\vec{x}). \end{cases}$$

Therefore, writing the inhomogeneous wave equation as the system,

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -1 \\ -c^2\partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}$$
$$\begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix},$$

letting $S(t)$ denote the solution operator for this evolution equation, and defining $S_1(t)$, $S_2(t)$ such that

$$S(t)\Phi = \begin{bmatrix} S_1(t)\Phi \\ S_2(t)\Phi \end{bmatrix},$$

we see that the solution of the inhomogeneous wave equation on $\mathbb{R}$ is given by

$$u(x,t) = S_1(t)\begin{bmatrix} \phi \\ \psi \end{bmatrix} + \int_0^t S_1(t-s)\begin{bmatrix} 0 \\ f(s) \end{bmatrix} ds.$$

We claim that we can use the same idea to solve the inhomogeneous wave equation on an interval $[0,l]$. We state this formally for the case of Dirichlet boundary conditions, but this can be extended more generally to any symmetric boundary conditions.

Claim 14. *Consider the inhomogeneous wave equation on an interval* (6.41). *Let $S_1(t)$ denote the solution operator for the homogeneous equation,*

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u(0,t) = 0 = u(l,t). \end{cases} \tag{6.42}$$

That is, let $S_1(t)$ be the operator such that the solution of (6.42) is given by

$$v(x,t) = S_1(t)\begin{bmatrix}\phi(x)\\ \psi(x)\end{bmatrix}.$$

Then the solution of (6.41) is given by

$$u(x,t) \equiv S_1(t)\begin{bmatrix}\phi\\ \psi\end{bmatrix} + \int_0^t S_1(t-s)\begin{bmatrix}0\\ f(s)\end{bmatrix}\,ds. \tag{6.43}$$

Proof. As shown by our earlier discussion of Duhamel's principle, defining $u(x,t)$ by (6.43), u will satisfy the PDE and the initial conditions. The only thing that remains to be verified is that u will satisfy the boundary conditions. Let

$$\Phi(x) \equiv \begin{bmatrix}\phi(x)\\ \psi(x)\end{bmatrix}.$$

By assumption, $S_1(t)\Phi$ satisfies (6.42) for all Φ. Therefore, $S_1(t)\Phi(0) = 0 = S_1(t)\Phi(l)$. Therefore,

$$\begin{aligned} u(0,t) &= S_1(t)\begin{bmatrix}\phi(0)\\ \psi(0)\end{bmatrix} + \int_0^t S_1(t-s)\begin{bmatrix}0\\ f(0,s)\end{bmatrix}\,ds \\ &= 0 + \int_0^t 0\,ds = 0. \end{aligned}$$

Similarly, $u(l,t) = 0$. □

Example 15. Solve the initial-value problem,

$$\begin{cases} u_{tt} - u_{xx} = f(x,t) & 0 < x < \pi \\ u(x,0) = \phi(x) & 0 < x < \pi \\ u_t(x,0) = \psi(x) & 0 < x < \pi \\ u(0,t) = 0 = u(\pi,t). \end{cases} \tag{6.44}$$

We know that the solution of the homogeneous equation,

$$\begin{cases} v_{tt} - v_{xx} = 0 & 0 < x < \pi \\ v(x,0) = \phi(x) & 0 < x < \pi \\ v_t(x,0) = \psi(x) & 0 < x < \pi \\ v(0,t) = 0 = v(\pi,t) \end{cases}$$

is given by

$$v(x,t) = \sum_{n=1}^{\infty}[A_n\cos(nt) + B_n\sin(nt)]\sin(nx),$$

where

$$\begin{aligned} A_n &= \frac{\langle\sin(nx),\phi(x)\rangle}{\langle\sin(nx),\sin(nx)\rangle} = \frac{2}{\pi}\int_0^\pi \sin(nx)\phi(x)\,dx \\ nB_n &= \frac{\langle\sin(nx),\psi(x)\rangle}{\langle\sin(nx),\sin(nx)\rangle} = \frac{2}{\pi}\int_0^\pi \sin(nx)\psi(x)\,dx \end{aligned}$$

for $n = 1, 2, \ldots$. Therefore, the solution operator associated with the homogeneous equation by

$$S_1(t)\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \sum_{n=1}^{\infty}[A_n\cos(nt) + B_n\sin(nt)]\sin(nx),$$

with A_n, B_n as defined above. Therefore,

$$S_1(t-s)\begin{bmatrix}0\\ f(s)\end{bmatrix} = \sum_{n=1}^{\infty}[C_n(s)\cos(n(t-s)) + D_n(s)\sin(n(t-s))]\sin(nx)$$

where

$$C_n(s) = 0$$
$$nD_n(s) = \frac{\langle\sin(nx), f(x,s)\rangle}{\langle\sin(nx),\sin(nx)\rangle} = \frac{2}{\pi}\int_0^{\pi}\sin(nx)f(x,s)\,dx$$

for $0 \le s \le t$, $n = 1, 2, \ldots$.

Therefore, the solution of (6.44) is given by

$$\boxed{u(x,t) = \sum_{n=1}^{\infty}[A_n\cos(nt) + B_n\sin(nt)]\sin(nx) + \int_0^t\sum_{n=1}^{\infty}D_n(s)\sin(n(t-s))\sin(nx)\,ds} \tag{6.45}$$

with $A_n, B_n, D_n(s)$ as defined above.

◇

6.5 Inhomogeneous Boundary Data

We now consider the wave equation with inhomogeneous boundary data,

$$\begin{cases} u_{tt} - c^2u_{xx} = f(x,t) & 0 < x < l\\ u(x,0) = \phi(x) & 0 < x < l\\ u_t(x,0) = \psi(x) & 0 < x < l\\ u(0,t) = g(t), u(l,t) = h(t). \end{cases}$$

Method of Shifting the Data

Ref: Strauss: Sec. 5.6.

Our plan is to introduce a new function $\mathcal{U}$ such that

$$\mathcal{U}(0,t) = g(t)$$
$$\mathcal{U}(l,t) = h(t).$$

Then, defining $v(x,t) = u(x,t) - \mathcal{U}(x,t)$, we will study the new initial-value problem which now has zero boundary data,

$$\begin{cases} v_{tt} - c^2v_{xx} = f(x,t) - \mathcal{U}_{tt} + c^2\mathcal{U}_{xx} & 0 < x < l\\ v(x,0) = \phi(x) - \mathcal{U}(x,0) & 0 < x < l\\ v_t(x,0) = \psi(x) - \mathcal{U}_t(x,0) & 0 < x < l\\ v(0,t) = 0 = v(l,t). \end{cases} \tag{6.46}$$

In order for $\mathcal{U}$ to satisfy the conditions stated, we define $\mathcal{U}$ as a linear function of x such that

$$\boxed{\mathcal{U}(x,t) = \left[\frac{h(t) - g(t)}{l}\right] x + g(t).}$$

Now $\mathcal{U}_{xx} = 0$. Therefore (6.46) becomes

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t) - \mathcal{U}_{tt} & 0 < x < l \\ v(x,0) = \phi(x) - \mathcal{U}(x,0) & 0 < x < l \\ v_t(x,0) = \psi(x) - \mathcal{U}_t(x,0) & 0 < x < l \\ v(0,t) = 0 = v(l,t). \end{cases} \tag{6.47}$$

This problem we can solve using Duhamel's principle discussed above.

Remark. If the boundary conditions g, h and the inhomogeneous term f do not depend on t, then the method of shifting the data is especially nice because we can immediately reduce the problem to a homogeneous IVP with homogeneous boundary conditions. In particular, consider

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x) & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u(0,t) = g, u(l,t) = h \end{cases}$$

Now considering (6.46), if we find $\mathcal{U}(x)$ such that

$$\begin{aligned} -c^2 \mathcal{U}_{xx} &= f(x) \\ \mathcal{U}(0) &= g \\ \mathcal{U}(l) &= h \end{aligned}$$

then, letting $v(x,t) = u(x,t) - \mathcal{U}(x)$, we see immediately that v will be a solution of

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & 0 < x < l \\ v(x,0) = \phi(x) - \mathcal{U}(x) & 0 < x < l \\ v_t(x,0) = \psi(x) & 0 < x < l \\ v(0,t) = 0 = v(l,t), \end{cases} \tag{6.48}$$

a homogeneous initial-value problem with Dirichlet boundary conditions.

6.6 Uniqueness

In the previous sections, we used the method of reflection and separation of variables to find a solution of the wave equation on an interval satisfying certain boundary conditions. We now prove that the solutions we found are in fact unique.

Claim 16. *Consider the initial-value problem for the inhomogeneous wave equation on an interval with Dirichlet boundary conditions,*

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u(0,t) = 0 = u(l,t). \end{cases} \tag{6.49}$$

There exists at most one (smooth) solution of (6.49).

Proof. Suppose there are two solutions u, v of (6.49). Let $w = u - v$. Therefore, w is a solution of

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0 & 0 < x < l \\ w(x,0) = 0 & 0 < x < l \\ w_t(x,0) = 0 & 0 < x < l \\ w(0,t) = 0 = w(l,t). \end{cases} \tag{6.50}$$

Define the energy function

$$E(t) = \frac{1}{2} \int_0^l w_t^2 + c^2 w_x^2 \, dx.$$

We see that

$$E(0) = \frac{1}{2} \int_0^l w_t^2(0,t) + c^2 w_x^2(0,t) \, dx = 0.$$

We claim that $E'(t) = 0$. Integrating by parts, we have

$$\begin{aligned} E'(t) &= \int_0^l w_t w_{tt} + c^2 w_x w_{xt} \, dx \\ &= \int_0^l w_t w_{tt} - c^2 w_{xx} w_t \, dx + c^2 w_x w_t \Big|_{x=0}^{x=l} \\ &= \int_0^l w_t [w_{tt} - c^2 w_{xx}] \, dx + 0, \end{aligned}$$

using the fact that $w(0,t) = 0 = w(l,t)$ for all t implies that $w_t(0,t) = 0 = w_t(l,t)$. As w is a solution of the homogeneous wave equation, we conclude that $E'(t) = 0$. Therefore, using the fact that $E(0) = 0$, we conclude that $E(t) = 0$. Then, using the fact that w is a smooth solution, we conclude that $w_x(x,t) = 0 = w_t(x,t)$. Therefore, $w(x,t) = C$ for some constant C. Using the fact that $w(x,0) = 0$, we conclude that $w(x,t) = 0$, and, therefore, $u(x,t) = v(x,t)$.

□