# 8 Hyperbolic Systems of First-Order Equations

*Ref: Evans, Sec. 7.3*

## 8.1 Definitions and Examples

Let $U : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^m$. Let $A_i(x,t)$ be an $m \times m$ matrix for $i = 1,\dots,n$. Let $F : \mathbb{R}^n \times (0,\infty) \to \mathbb{R}^m$. Consider the system

$$U_t + \sum_{i=1}^n A_i(x,t) U_{x_i} = F(x,t). \tag{8.1}$$

Fix $\xi \in \mathbb{R}^n$. Let

$$A(x,t;\xi) \equiv \sum_{i=1}^n A_i(x,t)\xi_i.$$

The system (8.1) is **hyperbolic** if $A(x,t;\xi)$ is diagonalizable for all $x,\xi \in \mathbb{R}^n, t > 0$. In particular, a system is hyperbolic if for all $x,\xi \in \mathbb{R}^n, t > 0$ the matrix $A(x,t;\xi)$ has $m$ real eigenvalues

$$\lambda_1(x,t;\xi) \le \lambda_2(x,t;\xi) \le \ldots \le \lambda_m(x,t;\xi)$$

corresponding to eigenvectors $\{r_i(x,t;\xi)\}_{i=1}^m$ which form a basis for $\mathbb{R}^m$.

There are two special cases of hyperbolicity which we now define.

1. If $A_i(x,t)$ is symmetric for $i = 1,\dots,n$, then $A(x,t;\xi)$ is symmetric for all $\xi \in \mathbb{R}^n$. Recall that if the $m \times m$ matrix $A(x,t;\xi)$ is symmetric, then it is diagonalizable. For the case when the matrices $A_i(x,t)$ are all symmetric, we say the system (8.1) is **symmetric hyperbolic**.

2. If $A(x,t;\xi)$ has $m$ real, *distinct* eigenvalues

$$\lambda_1(x,t;\xi) < \lambda_2(x,t;\xi) < \ldots < \lambda_m(x,t;\xi)$$

for all $x,\xi \in \mathbb{R}^n, t > 0$, then $A(x,t;\xi)$ is diagonalizable. In this case, we say the system (8.1) is **strictly hyperbolic**.

**Example 1.** The system

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

is strictly hyperbolic. ◇

**Example 2.** The system

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_x = \begin{bmatrix} f_1(x,t) \\ f_2(x,t) \end{bmatrix}$$

is symmetric hyperbolic. ◇

**Motivation.** Recall that all linear, constant-coefficient second-order hyperbolic equations can be written as

$$u_{tt} - \Delta u + \ldots = 0$$

through a change of variables, where "..." represents lower-order terms. One of the distinguishing features of the wave equation is that it has "wave-like" solutions. In particular, for the wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

the general solution is given by

$$u(x,t) = f(x+ct) + g(x-ct),$$

the sum of a wave moving to the right and a wave moving to the left. The functions $f(x+ct)$ and $g(x-ct)$ are known as **travelling waves**. More generally, for the wave equation in $\mathbb{R}^n$,

$$u_{tt} - c^2 \Delta u = 0 \qquad x \in \mathbb{R}^n, \tag{8.2}$$

for any (smooth) function $f$ and any $\xi \in \mathbb{R}^n$, $u(x,t) = f(\xi \cdot x - \sigma t)$ is a solution of (8.2) for $\sigma = \pm c|\xi|$. A solution of the form $f(\xi \cdot x - \sigma t)$ is known as a **plane wave solution**.

Motivated by the existence of plane wave solutions for the wave equation, we look for properties of the system (8.1) such that the equation will have plane wave solutions. The conditions under which plane wave solutions exist lead us to the definition of hyperbolicity given above.

First, we rewrite the wave equation as a system in the form of (8.1). First, consider the wave equation in one spatial dimension,

$$u_{tt} - u_{xx} = 0.$$

Let

$$U \equiv \begin{bmatrix} u_x \\ u_t \end{bmatrix}.$$

Then

$$U_t = \begin{bmatrix} u_{xt} \\ u_{tt} \end{bmatrix} = \begin{bmatrix} u_{tx} \\ u_{xx} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_t \end{bmatrix}_x = A_1 U_x$$

where

$$A_1 \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In general, for $x \in \mathbb{R}^n$, consider

$$u_{tt} - \Delta u = 0.$$

Let

$$U = \begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \\ u_t \end{bmatrix}.$$

Then

$$\begin{aligned}
U_t = \begin{bmatrix} u_{x_1 t} \\ \vdots \\ u_{x_n t} \\ u_{tt} \end{bmatrix} = \begin{bmatrix} u_{tx_1} \\ \vdots \\ u_{tx_n} \\ \sum_{i=1}^n u_{x_i x_i} \end{bmatrix} &= \begin{bmatrix} u_t \\ 0 \\ \vdots \\ 0 \\ u_{x_1} \end{bmatrix}_{x_1} + \begin{bmatrix} 0 \\ u_t \\ \vdots \\ 0 \\ u_{x_2} \end{bmatrix}_{x_2} + \ldots + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ u_t \\ u_{x_n} \end{bmatrix}_{x_n} \\
&= \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & & 0 \\ 0 & & \ddots & 0 \\ 1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \\ u_t \end{bmatrix}_{x_1} + \ldots + \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} u_{x_1} \\ \vdots \\ u_{x_n} \\ u_t \end{bmatrix}_{x_n} \\
&= \sum_{i=1}^n A_i U_{x_i}
\end{aligned}$$

where each $A_i$ is an $(n+1) \times (n+1)$ symmetric matrix whose entries $a^i_{jk}$ are given by

$$a^i_{jk} = \begin{cases} 1 & j = i, k = n+1; j = n+1, k = i \\ 0 & \text{otherwise.} \end{cases}$$

Now we claim that for $A_i$ as defined above, for each $\xi \in \mathbb{R}^n$, there are $m = n+1$ distinct plane wave solutions $U(x,t) = V(\xi \cdot x - \sigma t)$ of

$$U_t - \sum_{i=1}^n A_i U_{x_i} = 0.$$

In particular, define

$$A(\xi) = \sum_{i=1}^n A_i \xi_i.$$

Let $\lambda_i(\xi)$, $R_i(\xi)$ be the $i$th eigenvalue and corresponding eigenvector of $-A(\xi)$. Let $U(x,t) = f(\xi \cdot x - \lambda_i(\xi)t) R_i(\xi)$. Now

$$U_t = \lambda_i(\xi) f'(\xi \cdot x - \lambda_i(\xi)t) R_i(\xi)$$

and

$$U_{x_i} = \xi_i f'(\xi \cdot x - \lambda_i(\xi)t) R_i(\xi).$$

Therefore,

$$\begin{aligned} U_t - \sum_{i=1}^{n} A_i U_{x_i} &= f'(\xi \cdot x - \lambda_i(\xi)t) \left[ \lambda_i(\xi) R_i(\xi) - \sum_{i=1}^{n} A_i \xi_i R_i(\xi) \right] \\ &= f'(\xi \cdot x - \lambda_i(\xi)t) \left[ \lambda_i(\xi) R_i(\xi) - A(\xi) R_i(\xi) \right] = 0 \end{aligned}$$

because

$$-A(\xi) R_i(\xi) = \lambda_i(\xi) R_i(\xi).$$

Now notice that $A_i$ is a symmetric matrix for $i = 1, \ldots, n$. Therefore, $A(\xi) = \sum_{i=1}^{n} A_i \xi_i$ is an $m \times m$ symmetric matrix for each $\xi \in \mathbb{R}^n$. Consequently, $A(\xi)$ has $m$ real eigenvalues and $m$ linearly independent eigenvectors $R_i(\xi)$. Therefore, for each $\xi \in \mathbb{R}^n$ and each eigenvalue/eigenvector pair $\lambda_i(\xi), R_i(\xi)$, we get a distinct plane wave solution $U(x,t) = V(\xi \cdot x - \lambda_i(\xi)t)$.

We use this fact to define hyperbolicity for systems of the form (8.1). In particular, we want to find a condition on the system (8.1) under which there will be $m$ distinct plane wave solutions for each $\xi \in \mathbb{R}^n$. We look for a solution of (8.1) of the form $U(x,t) = V(\xi \cdot x - \sigma t)$. Plugging a function $U$ of this form into (8.1) with $F(x,t) \equiv 0$, we see this implies

$$-\sigma V' + \sum_{i=1}^{n} \xi_i A_i V' = 0. \tag{8.3}$$

Now if $\sum_{i=1}^{n} \xi_i A_i$ is an $m \times m$ diagonalizable matrix, then (8.3) will have $m$ solutions $V_1', \ldots, V_m'$. These solutions are the eigenvectors of $\sum_{i=1}^{n} \xi_i A_i$ which correspond to the $m$ eigenvalues $\sigma_1, \ldots, \sigma_m$. As a result, if $\sum_{i=1}^{n} \xi_i A_i$ is diagonalizable, then we have $m$ plane wave solutions of (8.1). This criteria gives us our definition for hyperbolicity described above.

## 8.2 Solving Hyperbolic Systems.

In this section, we will solve hyperbolic systems of the form

$$U_t + AU_x = F(x,t) \tag{8.4}$$

where $A$ is a constant-coefficient matrix. Note that if (8.4) is hyperbolic, then $A$ must be a diagonalizable matrix. Therefore, there exists an $m \times m$ invertible matrix $Q$ and an $m \times m$ diagonal matrix $\Lambda$ such that

$$Q^{-1} A Q = \Lambda.$$

In particular, $\Lambda$ is the diagonal matrix of eigenvalues and $Q$ is the matrix of eigenvectors. Therefore,

$$A = Q \Lambda Q^{-1}.$$

Substituting this into (8.4), our system becomes

$$U_t + Q \Lambda Q^{-1} U_x = F(x,t). \tag{8.5}$$

Now multiplying (8.5) by $Q^{-1}$, our system becomes

$$Q^{-1}U_t + \Lambda Q^{-1}U_x = Q^{-1}F(x,t).$$

Letting $V = Q^{-1}U$, we arrive at the decoupled system,

$$V_t + \Lambda V_x = Q^{-1}F(x,t).$$

*Remark:* If $A$ is symmetric, the eigenvectors may be chosen to be orthonormal, in which case $Q^{-1} = Q^T$.

**Example 3.** Find a solution to the initial-value problem

$$\begin{cases} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} u_1(x,0) \\ u_2(x,0) \end{bmatrix} = \begin{bmatrix} \sin x \\ \cos x \end{bmatrix}. \end{cases} \tag{8.6}$$

Here

$$A = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$$

is a symmetric matrix. Therefore, it has two real eigenvalues and its eigenvectors form an orthonormal basis for $\mathbb{R}^2$. In particular, $A$ can be diagonalized. The eigenvalue/eigenvector pairs are given by

$$\begin{aligned} \lambda_1 &= 5 \qquad \mathbf{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\ \lambda_2 &= -3 \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}. \end{aligned}$$

Therefore, $A = Q\Lambda Q^T$ where

$$\begin{aligned} Q &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ Q^T &= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \\ \Lambda &= \begin{bmatrix} 5 & 0 \\ 0 & -3 \end{bmatrix}. \end{aligned}$$

Our system can be rewritten as

$$Q^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \Lambda Q^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Letting $V = Q^T U$ our system becomes

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_t + \Lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_x = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} v_1(x,0) \\ v_2(x,0) \end{bmatrix} = Q^T \begin{bmatrix} u_1(x,0) \\ u_2(x,0) \end{bmatrix}.$$

That is, we have two separate transport equations,

$$\begin{cases} v_{1t} + 5v_{1x} = 0 \\ v_1(x,0) = \dfrac{1}{\sqrt{2}}(\sin x + \cos x). \end{cases}$$

and

$$\begin{cases} v_{2t} - 3v_{2x} = 0 \\ v_2(x,0) = \dfrac{1}{\sqrt{2}}(\sin x - \cos x). \end{cases}$$

Our solutions are given by

$$v_1(x,t) = \frac{1}{\sqrt{2}}(\sin(x-5t) + \cos(x-5t))$$
$$v_2(x,t) = \frac{1}{\sqrt{2}}(\sin(x+3t) - \cos(x+3t)).$$

Now $V = Q^T U$ implies $U = QV$. Therefore, our solution is given by

$$U(x,t) = \frac{1}{2}\begin{bmatrix} \sin(x-5t) + \cos(x-5t) + \sin(x+3t) + \cos(x+3t) \\ \sin(x-5t) + \cos(x-5t) - \sin(x+3t) - \cos(x+3t) \end{bmatrix}.$$

◇

*Remark.* Notice in the above example that the value of the solution $U$ at the point $(x_0, t_0)$ depends only on the values of the initial conditions at the points $x_0 - 5t_0$ and $x_0 + 3t_0$. In the next section, we prove that for a symmetric hyperbolic system of the form

$$U_t + \sum_{i=1}^{n} A_i U_{x_i} = 0$$

that the domain of dependence for a solution at the point $(x_0, t_0)$ is contained within the region

$$\{(x,t) : |x - x_0| \leq M(t_0 - t)\}$$

where $M$ is an upper bound on the eigenvalues of $A(\xi) = \sum_{i=1}^n A_i \xi_i$ over all $\xi \in \mathbb{R}^n$ such that $|\xi| = 1$; that is,

$$M \equiv \max_{\substack{i=1,\dots,m \\ |\xi|=1}} |\lambda_i(\xi)|,$$

where $\lambda_i(\xi)$ are the $m$ eigenvalues of $A(\xi)$.

## 8.3 Domain of Dependence for a Symmetric Hyperbolic System.

In the above example, we saw that the domain of dependence for a solution $U$ at the point $(x_0, t_0)$ is contained within the triangular region $\{(x,t) : x_0+5(t-t_0) \leq x \leq x_0-3(t-t_0), 0 \leq t \leq t_0\}$. More generally, for any symmetric hyperbolic system of the form

$$U_t + AU_x = F(x,t) \qquad x \in \mathbb{R},$$

the domain of dependence is contained within the region $\{(x,t) : x_0 + M(t-t_0) \leq x \leq x_0 - M(t-t_0), 0 \leq t \leq t_0\}$ where

$$M = \max_i |\lambda_i|$$

where $\lambda_i$ are the $m$ eigenvalues of $A$.

This idea extends to systems in higher dimensions in well. Consider the symmetric hyperbolic system,

$$U_t + \sum_{i=1}^n A_i U_{x_i} = 0 \qquad x \in \mathbb{R}^n$$

where each $A_i$ is an $m \times m$ symmetric matrix. Let

$$A(\xi) = \sum_{i=1}^n \xi_i A_i.$$

Let $\lambda_i(\xi)$, $i = 1, \dots, m$ be the $m$ eigenvalues of $A(\xi)$. Let

$$M = \max_{\substack{i=1,\dots,m \\ |\xi|=1}} |\lambda_i(\xi)|.$$

We claim that $M$ is the upper bound on the speed of waves in any direction. We state this more precisely in the following theorem.

First, we make some definitions. Let $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Let $0 \leq t_1 \leq t_0$. Let

$$\begin{aligned} B &\equiv \{x \in \mathbb{R}^n : |x - x_0| \leq Mt_0\} \\ C &\equiv \{x \in \mathbb{R}^n : |x - x_0| \leq M(t_0 - t_1)\} \\ S &\equiv \{(x,t) : 0 \leq t \leq t_1, |x - x_0| = M|t_0 - t|\}. \end{aligned}$$

**Theorem 4.** *Let $(x_0, t_0) \in \mathbb{R}^n \times (0, \infty)$. Let $B, C$ be as defined above. Assume $U$ is a solution of*

$$U_t + \sum_{i=1}^n A_i U_{x_i} = 0 \tag{8.7}$$

*where each $A_i$ is an $m \times m$ constant-coefficient, symmetric matrix. If $|U| \equiv 0$ on $B$, then $|U| \equiv 0$ on $C$.*

*Proof.* Let $\Omega$ be the region bounded by $B, C$ and $S$. Let $\partial\Omega = B \cup C \cup S$.

Multiplying (8.7) by $U^T$ and integrating over $\Omega$, we have

$$\int_\Omega U^T(U_t + \sum_{i=1}^n A_i U_{x_i})\,dx\,dt = 0.$$

First, we note that

$$U^T U_t = \frac{1}{2}\partial_t |U|^2.$$

Second, we have

$$U^T \sum_{i=1}^n A_i U_{x_i} = \frac{1}{2}\sum_{i=1}^n (U \cdot A_i U)_{x_i},$$

using the fact that each $A_i$ is symmetric. Then, by the divergence theorem, we have

$$\int_\Omega \frac{1}{2}\partial_t |U|^2\,dx\,dt = \frac{1}{2}\int_{\partial\Omega} |U|^2 \nu_t\,dS$$

where $\nu = (\nu_1, \dots, \nu_n, \nu_t)$ is the outward unit normal on $\partial\Omega$. Now

$$\frac{1}{2}\int_{\partial\Omega} |U|^2 \nu_t\,dS = \frac{1}{2}\int_C |U|^2\,dx - \frac{1}{2}\int_B |U|^2\,dx + \frac{1}{2}\int_S |U|^2 \nu_t\,dS.$$

On $S$,

$$\nu = \frac{(x_1 - x_{01}, \dots, x_n - x_{0n}, M^2(t - t_0))}{(t_0 - t)M\sqrt{1 + M^2}}.$$

Therefore,

$$\nu_t = \frac{M}{\sqrt{1 + M^2}}.$$

Therefore,

$$\frac{1}{2}\int_{\partial\Omega} |U|^2 \nu_t\,dS = \frac{1}{2}\int_C |U|^2\,dx - \frac{1}{2}\int_B |U|^2\,dx + \frac{1}{2}\int_S |U|^2 \frac{M}{\sqrt{1 + M^2}}\,dS.$$

Similarly, we use the divergence theorem on our other term,

$$\begin{aligned}\int_\Omega \frac{1}{2}\sum_{i=1}^n (U \cdot A_i U)_{x_i}\,dx\,dt &= \frac{1}{2}\int_S \sum_{i=1}^n (U \cdot A_i U)\nu_i\,dS \\ &= \frac{1}{2}\int_S \sum_{i=1}^n (U \cdot A_i U)\frac{(x_i - x_{0i})}{(t_0 - t)M\sqrt{1 + M^2}}\,dS.\end{aligned}$$

Let

$$\xi_i = \frac{(x_i - x_{0i})}{(t_0 - t)M}.$$

Then, we have $|\xi| = 1$. And, therefore,

$$\begin{aligned}\int_\Omega \frac{1}{2}\sum_{i=1}^n (U \cdot A_i U)_{x_i}\,dx\,dt &= \frac{1}{2\sqrt{1+M^2}}\int_S \sum_{i=1}^n (U \cdot A_i \xi_i U)\,dS \\ &= \frac{1}{2\sqrt{1+M^2}}\int_S (U \cdot A(\xi)U)\,dS.\end{aligned}$$

Therefore, we have

$$\begin{aligned}0 &= \int_\Omega U^T (U_t + \sum_{i=1}^n A_i U_{x_i})\,dx\,dt \\ &= \frac{1}{2}\int_C |U|^2\,dx - \frac{1}{2}\int_B |U|^2\,dx + \frac{1}{2}\int_S |U|^2 \frac{M}{\sqrt{1+M^2}}\,dS + \frac{1}{2\sqrt{1+M^2}}\int_S (U \cdot A(\xi)U)\,dS.\end{aligned}$$

But,

$$\begin{aligned}|U|^2 M + U \cdot A(\xi)U &= U \cdot (MI + A(\xi))U \\ &\geq 0,\end{aligned}$$

as the eigenvalues of $A(\xi) \leq M$.

Therefore,

$$\frac{1}{2}\int_C |U|^2\,dx \leq \frac{1}{2}\int_B |U|^2\,dx.$$

Therefore, if $|U| \equiv 0$ on $B$, then $|U| \equiv 0$ on $C$.

□

Now we prove uniqueness of solutions to symmetric hyperbolic systems.

**Theorem 5.** (Uniqueness) *Consider the symmetric hyperbolic system,*

$$\begin{cases} U_t + \sum_{i=1}^n A_i U_{x_i} = F(x,t) & x \in \mathbb{R}^n \\ U(x,0) = \Phi(x) \end{cases}$$

*where the initial data $\Phi(x)$ has compact support. Then there exists at most one (smooth) solution.*

*Proof.* Suppose there are two smooth solutions $U_1$ and $U_2$ with the same initial data $\Phi$. Let $W(x,t) = U_1(x,t) - U_2(x,t)$. We know that $W(x,0) = 0$. We claim that $W(x,t) \equiv 0$. Define the energy function,

$$E(t) \equiv \frac{1}{2}\int_{\mathbb{R}^n} |W|^2\,dx.$$

We know $E(0) = 0$. We claim $E(t) = 0$, and, therefore, $W(x,t) \equiv 0$. We will show that $E(t) = 0$ by showing that $E'(t) = 0$.

$$\begin{aligned}
E'(t) &= \int_{\mathbb{R}^n} W \cdot W_t \, dx \\
&= -\int_{\mathbb{R}^n} W \cdot \sum_{i=1}^{n} A_i W_{x_i} \\
&= -\frac{1}{2} \int_{\mathbb{R}^n} \sum_{i=1}^{n} (W \cdot A_i W)_{x_i} \, dx \\
&= 0
\end{aligned}$$

using the fact that $\Phi$ has compact support implies $W$ has compact support. (By the domain of dependence results we showed above.) Therefore, $E'(t) = 0$, which implies for $E(0) = 0$, then $E(t) = 0$ for any time $t$. Therefore, $|W| \equiv 0$ which implies $U_1 \equiv U_2$ (assuming $U_1, U_2$ are smooth).

□