Math 220A
Practice Midterm Solutions - Fall 2002

1. Classify the following in terms of degree of nonlinearity:

   (a) $u_t + e^u u_x = x^2$ **Answer:** quasilinear

   (b) $u_t + x^2 u_x = e^u$ **Answer:** semilinear

   (c) $x^3 u_t + u_x^2 = 1$ **Answer:** fully nonlinear

   (d) $u_t + x^3 u_x = \sin(x^2)$ **Answer:** linear

   (e) $u_{xt} + \left[\frac{u^2}{2}\right]_x = \cos(u_x)$ **Answer:** semilinear

2. Find the unique weak solution of

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

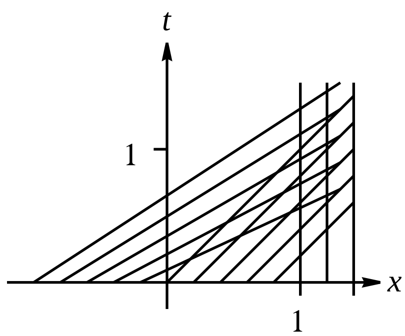
$$\phi(x) = \begin{cases} 2 & x < 0 \\ 1 & 0 < x < 1 \\ 0 & x > 1 \end{cases}$$

which satisfies the entropy condition.

Answer: The characteristic ODE are given by

$$\begin{cases} \dfrac{dt}{ds} = 1 & t(r,0) = 0 \\ \dfrac{dx}{ds} = z & x(r,0) = r \\ \dfrac{dz}{ds} = 0 & z(r,0) = \phi(r) \end{cases}$$

which implies the projected characteristics are given by $x = \phi(r)t + r$. For $r < 0$, our projected characteristics are given by $x = 2t + r$. For $0 < r < 1$, our projected characteristics are given by $x = t + r$. For $r > 1$, our projected characteristics are given by $x = r$.



Therefore, we need to put in two shock curves.

First, between $u^- = 2$ and $u^+ = 1$, our shock curve $x = \xi_1(t)$ needs to satisfy

$$\xi_1'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{2^2}{2} - \frac{1^2}{2}}{2-1} = \frac{3}{2}.$$

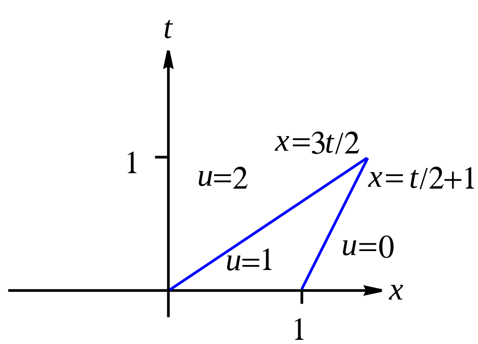
In addition, this curve needs to pass through the point $(0, 0)$. Therefore, $\xi_1(t) = \frac{3}{2}t$.

Second, between $u^- = 1$ and $u^+ = 0$, we need the shock curve $x = \xi_2(t)$ to satisfy

$$\xi_2'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{1^2}{2} - \frac{0^2}{2}}{1-0} = \frac{1}{2}.$$

In addition, this curve needs to pass through the point $(1, 0)$. Therefore, $\xi_2(t) = \frac{1}{2}t + 1$. Therefore, our solution is given by

$$u(x,t) = \begin{cases} 2 & x < \frac{3}{2}t \\ 1 & \frac{3}{2}t < x < \frac{1}{2}t + 1 \\ 0 & x > \frac{1}{2}t + 1. \end{cases}$$



However, at $t = 1$, the two shock curves intersect. Therefore, the above solution is only valid for $0 \leq t \leq 1$. Consequently, we need to define a new shock curve $x = \xi_3(t)$

starting from the point $(3/2, 1)$ which satisfies the Rankine-Hugoniot jump condition where $u^- = 2$ and $u^+ = 0$. That is,

$$\xi_3'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{2^2}{2} - \frac{0^2}{2}}{2 - 0} = 1.$$

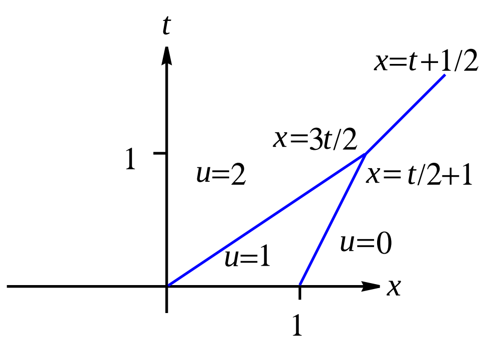
Therefore, $\xi_3(t) = t + \frac{1}{2}$.

In summary, for $\boxed{0 \le t \le 1}$ our solution is given by

$$\boxed{u(x,t) = \begin{cases} 2 & x < \frac{3}{2}t \\ 1 & \frac{3}{2}t < x < \frac{1}{2}t + 1 \\ 0 & x > \frac{1}{2}t + 1. \end{cases}}$$

While for $\boxed{t \ge 1}$, our solution is given by

$$\boxed{u(x,t) = \begin{cases} 2 & x < t + \frac{1}{2} \\ 0 & x > t + \frac{1}{2} \end{cases}}$$



3. Solve

$$\begin{cases} u_t + u_x^2 = 0 \\ u(x,0) = -x^2 \end{cases}$$

Find the time T for which $|u| \to +\infty$ as $t \to T$.

Answer: Writing our equation as $F(x, t, z, p, q) = p^2 + q$, our characteristic ODE are given by

$$\begin{cases} \dfrac{dx}{ds} = 2p & x(r,0) = r \\ \dfrac{dt}{ds} = 1 & t(r,0) = 0 \\ \dfrac{dz}{ds} = p^2 & z(r,0) = -r^2 \\ \dfrac{dp}{ds} = 0 & p(r,0) = -2r \\ \dfrac{dq}{ds} = 0 & q(r,0) = -4r^2. \end{cases}$$

Note the initial conditions for p and q, denoted $p(r,0) = \psi_1(r)$ and $q(r,0) = \psi_2(r)$ come from the two equations

$$\psi_1^2 + \psi_2 = 0$$
$$\phi' = \psi_1\gamma_1' + \psi_2\gamma_2'.$$

Now solving this system of ODEs, we have

$$\begin{cases} x = -4rs + r \\ t = s \\ z = 4r^2 s - r^2 \\ p = -2r \\ q = -4r^2 \end{cases}$$

Solving for r and s in terms of x and t, we have

$$\boxed{u(x,t) = 4\left(\frac{x}{1-4t}\right)^2 t - \left(\frac{x}{1-4t}\right)^2.}$$

The solution is finite up until $T = 1/4$.

4. Solve

$$\begin{cases} u_x + xu_y - 4u = 0 \\ u(1,y) = y^2. \end{cases}$$

Answer: Our characteristic ODE are given by

$$\begin{cases} \dfrac{dx}{ds} = 1 & x(r,0) = 1 \\ \dfrac{dy}{ds} = x & y(r,0) = r \\ \dfrac{dz}{ds} = 4z & z(r,0) = r^2. \end{cases}$$

Solving the characteristic ODE, we have

$$\begin{cases} x = s + 1 \\ y = s^2/2 + s + r \\ z = r^2 e^{4s}. \end{cases}$$

Solving for r and s in terms of x and y, we see our solution is given by

$$\boxed{u(x,y) = \left[y - \frac{(x-1)^2}{2} - (x-1)\right]^2 e^{4(x-1)}.}$$

5. Find the unique weak solution of

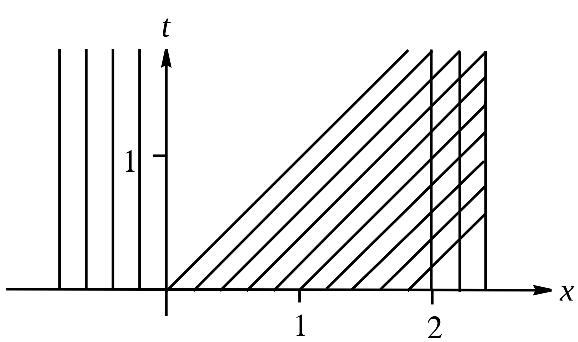
$$\begin{cases} u_t + u^2 u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 < x < 2 \\ 0 & x > 2 \end{cases}$$

which satisfies the Oleinik entropy condition.

Answer: As in problem 2, the projected characteristics are given by $x = \phi(r)t + r$. For $r < 0$, $\phi(r) = 0$ implies the proj. chars. are given by $x = r$ for $r < 0$. For $0 < r < 2$, $\phi(r) = 1$ implies the proj. chars. are given by $x = t + r$ for $0 < r < 2$. For $r > 1$, $\phi(r) = 1$ implies the proj. chars. are given by $x = r$ for $r > 2$.



We put in a rarefaction wave in the open wedge and introduce a shock curve where the projected characteristics cross. The rarefaction wave is given by $G(x/t) = (f')^{-1}(x/t) = \sqrt{x/t}$. The shock curve $x = \xi_1(t)$ must satisfy the R-H jump condition,

$$\xi_1'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{1^3}{3} - \frac{0^3}{3}}{1 - 0} = \frac{1}{3}.$$

Notice that the Oleinik entropy condition is satisfied because the chord connecting $(u^-, f(u^-)) = (1, 1/3)$ and $(u^+, f(u^+)) = (0, 0)$ lies above the graph of the function f.

Combining the rarefaction wave and shock curve defined above, we see our solution is given by

$$u(x,t) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{x}{t}} & 0 < x < t \\ 1 & t < x < \frac{1}{3}t + 2 \\ 0 & x > \frac{1}{3}t + 2 \end{cases}$$

Now at $t = 3$, the rarefaction wave hits the shock curve. Therefore, we need to introduce a new shock curve. The new shock curve $x = \xi_2'(t)$ must satisfy

$$\xi_2'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{1}{3}\left(\sqrt{\frac{x}{t}}\right)^3 - \frac{0^3}{3}}{\sqrt{\frac{x}{t}} - 0} = \frac{x}{3t}.$$

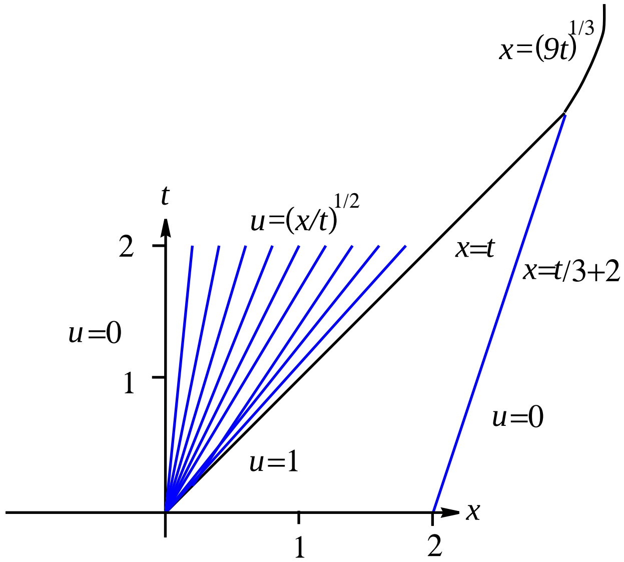
In addition, this curve must pass through the point $(x,t) = (3,3)$. Therefore, this curve is given by $x = (9t)^{1/3}$. Again, notice that the Oleinik entropy condition is satisfied along this shock curve.

In summary, our solution is given as follows. For $\boxed{0 \leq t \leq 3}$,

$$\boxed{u(x,t) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{x}{t}} & 0 < x < t \\ 1 & t < x < \frac{1}{3}t + 2 \\ 0 & x > \frac{1}{3}t + 2 \end{cases}}$$

While for $\boxed{t \geq 3}$, our solution is given by

$$\boxed{u(x,t) = \begin{cases} 0 & x < 0 \\ \sqrt{\frac{x}{t}} & 0 < x < (9t)^{1/3} \\ 0 & x > (9t)^{1/3}. \end{cases}}$$



6. Let f, g be C^∞ functions. Consider the initial value problem

$$\begin{cases} [g(u)]_t + [f(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

(a) Give a definition for a strong solution of this initial value problem.

Answer: We say u is a strong solution of the initial-value problem above if u is a continuously differentiable function which satisfies the equation $[g(u)]_t + [f(u)]_x = 0$ at each point (x,t) and $u(x,0) = \phi(x)$.

(b) Give a definition for a weak solution of this initial value problem.

Answer: We say u is a weak solution of the initial-value problem above if u satisfies

$$\int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x] \, dx \, dt + \int_{-\infty}^\infty g(\phi(x))v(x,0) \, dx = 0$$

for all smooth functions v with compact support.

(c) Prove that any strong solution is in fact a weak solution.

Answer: If u is strong, then

$$[g(u)]_t + [f(u)]_x = 0.$$

Now multiplying this equation by a smooth function v with compact support and integrating over $\mathbb{R} \times [0, \infty)$, we have

$$\int_0^\infty \int_{-\infty}^\infty \{[g(u)]_t + [f(u)]_x\} \, v \, dx \, dt = 0.$$

Integrating by parts, we have

$$-\int_0^\infty \int_{-\infty}^\infty g(u)v_t \, dx \, dt + \int_{-\infty}^\infty g(u)v \, dx\bigg|_{t=0}^{t=\infty} \\ - \int_0^\infty \int_{-\infty}^\infty f(u)v_x \, dx \, dt + \int_0^\infty f(u)v \, dt\bigg|_{x=-\infty}^{x=\infty} = 0.$$

Using the fact that v has compact support, three of the boundary terms vanish. Therefore, we are left with

$$-\int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x] \, dx \, dt - \int_{-\infty}^\infty g(\phi(x))v(x,0) \, dx = 0,$$

as desired.

7. (a) Find the general solution of

$$u_{tt} + 2u_{xt} - 3u_{xx} = 0.$$

Answer: We rewrite our equation as

$$(\partial_t + 3\partial_x)(\partial_t - \partial_x)u = 0.$$

Then we introduce new variables ξ, η such that

$$\partial_\xi = \partial_t + 3\partial_x$$
$$\partial_\eta = \partial_t - \partial_x.$$

Therefore,

$$\xi = \frac{1}{4}(x + t)$$
$$\eta = -\frac{1}{4}(x - 3t).$$

With this change of variables, we have

$$u_{\xi\eta} = 0.$$

Therefore, the general solution is given by

$$\boxed{u(x,t) = f(x+t) + g(x-3t).}$$

(b) Find the solution of the initial-value problem,

$$\begin{cases} u_{tt} + 2u_{xt} - 3u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

Answer: Our general solution is given by

$$u(x,t) = f(x+t) + g(x-3t).$$

We want

$$\begin{aligned} u(x,0) &= f(x) + g(x) = \phi(x) \\ u_t(x,0) &= f'(x) - 3g'(x) = \psi(x). \end{aligned}$$

Solving these equations for f and g, we arrive at the following solution for our initial-value problem,

$$\boxed{u(x,t) = \frac{3}{4}\phi(x+t) + \frac{1}{4}\phi(x-3t) + \frac{1}{4}\int_{x-3t}^{x+t} \psi(y)\,dy.}$$