Math 220A - Second Practice Midterm Exam Solutions - Fall 2002

1. Classify the following in terms of degree of nonlinearity:

   (a) $u_t^2 + x^2 u_{xt} = \sin(u)$ **Answer:** semilinear

   (b) $u_x + [u^3]_y = x^2 + y^2$ **Answer:** quasilinear

   (c) $[e^u]_x + u^2 u_y = 0$ **Answer:** quasilinear

   (d) $[x^3 u]_x + y^3 u = \sin(x^2 + y^2)$ **Answer:** linear

   (e) $[u_x^3]_t + e^{u_{xt}} = 0$ **Answer:** fully nonlinear

2. Solve

$$\begin{cases} u_t + u_x^2 = t \\ u(x,0) = x. \end{cases}$$

Answer: Let $F(x,t,z,p,q) = q + p^2 - t$. Our characteristic equations are given by

$$\begin{aligned} \frac{dx}{ds} &= F_p = 2p \\ \frac{dt}{ds} &= F_q = 1 \\ \frac{dz}{ds} &= pF_p + qF_q = 2p^2 + q \\ \frac{dp}{ds} &= -F_x - pF_z = 0 \\ \frac{dq}{ds} &= -F_t - qF_z = 1 \end{aligned}$$

with initial conditions

$$\begin{aligned} x(r,0) &= \gamma_1(r) = r \\ t(r,0) &= \gamma_2(r) = 0 \\ z(r,0) &= \phi(r) = r \\ p(r,0) &= \psi_1(r) \\ q(r,0) &= \psi_2(r) \end{aligned}$$

where ψ_1, ψ_2 satisfy

$$\begin{aligned} \phi'(r) &= \psi_1 \gamma_1' + \psi_2 \gamma_2' \\ \psi_2 + \psi_1^2 - \gamma_2 &= 0. \end{aligned}$$

The first equation implies $\psi_1 = 1$. The second equation implies $\psi_2 = -1$. Therefore the solutions of our characteristic equations are given as follows,

$$\begin{aligned} x(r,s) &= 2s + r \\ t(r,s) &= s \\ z(r,s) &= \frac{s^2}{2} + s + r \\ p(r,s) &= 1 \\ q(r,s) &= s - 1. \end{aligned}$$

Therefore, our solution is given by

$$u(x,t) = z(r(x,t), s(x,t)) = \frac{t^2}{2} + t + x - 2t,$$

or

$$\boxed{u(x,t) = \frac{t^2}{2} + x - t.}$$

3. Solve

$$\begin{cases} u_{tt} + 3u_{xt} - 10u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

by reducing the hyperbolic equation to two first-order transport equations. That is, reduce to the system

$$\begin{aligned} (\partial_t + 5\partial_x)v &= 0 \\ (\partial_t - 2\partial_x)u &= v \end{aligned}$$

with appropriate initial conditions. Then solve these first-order equations using the method of characteristics.

Answer: We write our equation

$$u_{tt} + 3u_{xt} - 10u_{xx} = 0$$

as

$$(\partial_t + 5\partial_x)(\partial_t - 2\partial_x)u = 0.$$

Now letting

$$v = (\partial_t - 2\partial_x)u,$$

our equation can be written as two transport equations,

$$\begin{aligned} u_t - 2u_x &= v \\ v_t + 5v_x &= 0. \end{aligned}$$

Now using the fact that $v = u_t - 2u_x$, we see that $v(x, 0) = \psi(x) - 2\phi'(x)$. Therefore, we must first solve the initial-value problem

$$\begin{aligned} &v_t + 5v_x = 0 \\ &v(x, 0) = \psi(x) - 2\phi'(x). \end{aligned}$$

We know solutions of the transport equation are given by

$$v(x, t) = f(x - 5t).$$

Combining this with our initial condition, we have

$$v(x, t) = \psi(x - 5t) - 2\phi'(x - 5t).$$

Next, we solve

$$\begin{aligned} &u_t - 2u_x = \psi(x - 5t) - 2\phi'(x - 5t) \\ &u(x, 0) = \phi(x) \end{aligned}$$

Our characteristic ODE are given by

$$\begin{aligned} &\frac{dx}{ds} = -2 \\ &\frac{dt}{ds} = 1 \\ &\frac{dz}{ds} = \psi(x(s) - 5t(s)) - 2\phi'(x(s) - 5t(s)) \end{aligned}$$

subject to the initial conditions

$$\begin{aligned} &x(r, 0) = r \\ &t(r, 0) = 0 \\ &z(r, 0) = \phi(r). \end{aligned}$$

The solutions of this system of ODEs is given by

$$\begin{aligned} x(r, s) &= -2s + r \\ t(r, s) &= s \\ z(r, s) &= \int_0^s \psi(-2s' + r - 5s')\, ds' - 2\int_0^2 \phi'(-2s' + r - 5s')\, ds' + \phi(r) \\ &= \int_0^s [\psi(-7s' + r) - 2\phi'(-7s' + r)]\, ds' + \phi(r) \\ &= -\frac{1}{7}\int_r^{-7s+r} [\psi(y) - 2\phi'(y)]\, dy + \phi(r). \end{aligned}$$

Therefore, our solution u is given by

$$u(x,t) = z(r(x,t), s(x,t)) = -\frac{1}{7}\int_{x+2t}^{x-5t} [\psi(y) - 2\phi'(y)]\, dy + \phi(x+2t)$$

which implies

$$\boxed{u(x,t) = \frac{5}{7}\phi(x+2t) + \frac{2}{7}\phi(x-5t) + \frac{1}{7}\int_{x-5t}^{x+2t} \psi(y)\, dy.}$$

4. Find the unique, weak solution of the following which satisfies the entropy condition,

$$\begin{cases} u_t - (\sin(u))_x = 0 & t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

in each of the two cases below.

Answer: First, our characteristic ODEs are given by

$$\frac{dt}{ds} = 0$$
$$\frac{dx}{ds} = -\cos(z)$$
$$\frac{dz}{ds} = 0$$

with initial conditions

$$t(r,0) = 0$$
$$x(r,0) = r$$
$$z(r,0) = \phi(r).$$

Therefore, our projected characteristics are given by

$$x = -\cos(\phi(r))t + r.$$

(a)

$$\phi(x) = \begin{cases} 0 & x < 0 \\ \pi & x > 0. \end{cases}$$

In this case, the characteristics are given by

$$x = -t + r \qquad r < 0$$
$$x = t + r \qquad r > 0.$$

Therefore, we need to fill in the open wedge with a rarefaction wave,

$$G(x/t) = (f')^{-1}(x/t) = \arccos(-x/t).$$

Therefore, our solution is given by

$$\boxed{u(x,t) = \begin{cases} 0 & x < -t \\ \arccos(-x/t) & -t < x < t \\ \pi & x > t. \end{cases}}$$

(b)

$$\phi(x) = \begin{cases} \pi & x < 0 \\ 0 & x > 0. \end{cases}$$

In this case, the characteristics are given by

$$\begin{aligned} x &= t + r & r < 0 \\ x &= -t + r & r > 0. \end{aligned}$$

Therefore, our projected characteristics intersect. We introduce a shock curve. This curve $x = \xi(t)$ must satisfy

$$\xi'(t) = \frac{[f(u)]}{[u]} = \frac{-\sin(\pi) + \sin(0)}{\pi - 0} = 0,$$

and pass through the origin. Therefore, the curve is $x = 0$. Therefore, our solution is given by

$$\boxed{u(x,t) = \begin{cases} \pi & x < 0 \\ 0 & x > 0. \end{cases}}$$

5. We say u is a weak solution of

$$(*) \begin{cases} [g(u)]_t + [f(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

if u satisfies

$$\int_0^\infty \int_{-\infty}^\infty g(u)v_t + f(u)v_x \, dx \, dt + \int_{-\infty}^\infty \phi(x)v(x,0) \, dx = 0$$

for all $v \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support. Suppose u is a weak solution of (*) such that u has a jump discontinuity across the curve $x = \xi(t)$, but u is smooth on either side of the curve $x = \xi(t)$. Let $u^-(x,t)$ be the value of u to the left of the curve and $u^+(x,t)$ be the value of u to the right of the curve. Prove that u must satisfy the condition

$$\frac{[f(u)]}{[g(u)]} = \xi'(t)$$

across the curve of discontinuity, where

$$\begin{aligned} [f(u)] &= f(u^-) - f(u^+) \\ [g(u)] &= g(u^-) - g(u^+). \end{aligned}$$

Answer: If u is a weak solution of (*), then

$$\int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x]\,dx\,dt + \int_{-\infty}^\infty g(\phi(x))v(x,0)\,dx = 0$$

for all smooth functions $v \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support. Let v be a smooth function such that $v(x,0) = 0$, and break up the first integral into the regions Ω^-, Ω^+ where

$$\Omega^- \equiv \{(x,t) : 0 < t < \infty,\ -\infty < x < \xi(t)\}$$
$$\Omega^+ \equiv \{(x,t) : 0 < t < \infty,\ \xi(t) < x < +\infty\}.$$

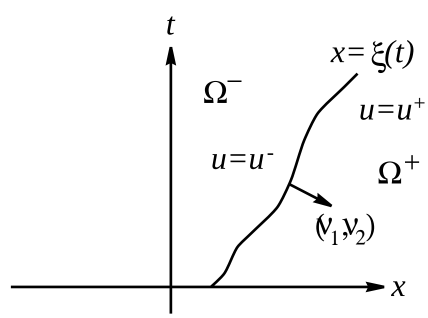
Therefore,

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x]\,dx\,dt + \int_{-\infty}^\infty g(\phi(x))v(x,0)\,dx \\ &= \iint_{\Omega^-} [g(u)v_t + f(u)v_x]\,dx\,dt + \iint_{\Omega^+} [g(u)v_t + f(u)v_x]\,dx\,dt. \end{aligned}$$

Combining the Divergence Theorem with the fact that v has compact support and $v(x,0) = 0$, we have

$$\begin{aligned} \iint_{\Omega^-} [g(u)v_t + f(u)v_x]\,dx\,dt = &- \iint_{\Omega^-} [(g(u))_t + (f(u))_x]v\,dx\,dt \\ &+ \int_{x=\xi(t)} [g(u^-)v\nu_2 + f(u^-)v\nu_1]\,ds \end{aligned}$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal to Ω^-.



Similarly, we see that

$$\begin{aligned} \iint_{\Omega^+} [g(u)v_t + f(u)v_x]\,dx\,dt = &- \iint_{\Omega^+} [(g(u))_t + (f(u))_x]v\,dx\,dt \\ &- \int_{x=\xi(t)} [g(u^+)v\nu_2 + f(u^+)v\nu_1]\,ds. \end{aligned}$$

By assumption, u is a weak solution of

$$[g(u)]_t + [f(u)]_x = 0$$

and u is smooth on either side of $x = \xi(t)$. Therefore, u is a strong solution on either side of the curve of discontinuity. Consequently, we see that

$$\iint_{\Omega^-} [(g(u))_t + (f(u))_x] v \, dx \, dt = 0 = \iint_{\Omega^+} [(g(u))_t + (f(u))_x] v \, dx \, dt.$$

Combining these facts, we see that

$$\int_{x=\xi(t)} [g(u^-) v \nu_2 + f(u^-) v \nu_1] \, ds - \int_{x=\xi(t)} [g(u^+) v \nu_2 + f(u^+) v \nu_1] \, ds = 0.$$

Since this is true for all smooth functions v, we have

$$g(u^-)\nu_2 + f(u^-)\nu_1 = g(u^+)\nu_2 + f(u^+)\nu_1,$$

which implies

$$\frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = -\frac{\nu_2}{\nu_1}.$$

Now the curve $x = \xi(t)$ has slope given by the negative reciprocal of the normal to the curve; that is,

$$\frac{dt}{dx} = \frac{1}{\xi'(t)} = -\frac{\nu_1}{\nu_2}.$$

Therefore,

$$\xi'(t) = -\frac{\nu_2}{\nu_1} = \frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = \frac{[f(u)]}{[g(u)]},$$

as claimed.