Math 220A **Second Practice Midterm Exam** **Fall 2002**

1. Classify the following in terms of degree of nonlinearity:

   (a) $u_t^2 + x^2 u_{xt} = \sin(u)$

   (b) $u_x + [u^3]_y = x^2 + y^2$

   (c) $[e^u]_x + u^2 u_y = 0$

   (d) $[x^3 u]_x + y^3 u = \sin(x^2 + y^2)$

   (e) $[u_x^3]_t + e^{u_{xt}} = 0$

2. Solve

$$\begin{cases} u_t + u_x^2 = t \\ u(x,0) = x. \end{cases}$$

3. Solve

$$\begin{cases} u_{tt} + 3u_{xt} - 10u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

   by reducing the hyperbolic equation to two first-order transport equations. That is, reduce to the system

$$(\partial_t + 5\partial_x)v = 0$$
$$(\partial_t - 2\partial_x)u = v$$

   with appropriate initial conditions. Then solve these first-order equations using the method of characteristics.

4. Find the unique, weak solution of the following which satisfies the entropy condition,

$$\begin{cases} u_t - (\sin(u))_x = 0 \qquad t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

   in each of the two cases below:

   (a)

$$\phi(x) = \begin{cases} 0 & x < 0 \\ \pi & x > 0. \end{cases}$$

   (b)

$$\phi(x) = \begin{cases} \pi & x < 0 \\ 0 & x > 0. \end{cases}$$

5. We say u is a weak solution of

$$(*) \begin{cases} [g(u)]_t + [f(u)]_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

if u satisfies

$$\int_0^\infty \int_{-\infty}^\infty g(u)v_t + f(u)v_x \, dx \, dt + \int_{-\infty}^\infty \phi(x)v(x,0) \, dx = 0$$

for all $v \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support. Suppose u is a weak solution of (*) such that u has a jump discontinuity across the curve $x = \xi(t)$, but u is smooth on either side of the curve $x = \xi(t)$. Let $u^-(x,t)$ be the value of u to the left of the curve and $u^+(x,t)$ be the value of u to the right of the curve. Prove that u must satisfy the condition

$$\frac{[f(u)]}{[g(u)]} = \xi'(t)$$

across the curve of discontinuity, where

$$[f(u)] = f(u^-) - f(u^+)$$
$$[g(u)] = g(u^-) - g(u^+).$$