Math 220A Practice Final Exam 2 Solutions Fall 2002

1. Solve the following initial-value problem,

$$\begin{cases} u_t + u_x + u^2 = 0 \\ u(x,0) = x. \end{cases}$$

Answer: The characteristic equations are given by

$$\begin{aligned} \frac{dt}{ds} &= 1 \qquad t(r,0) = 0 \\ \frac{dx}{ds} &= 1 \qquad x(r,0) = r \\ \frac{dz}{ds} &= -z^2 \qquad z(r,0) = r \end{aligned}$$

Therefore,

$$\begin{aligned} t &= s \\ x &= s + r \\ \frac{1}{z} &= t + \frac{1}{r}. \end{aligned}$$

Our solution is given by

$$\boxed{u(x,t) = \frac{x-t}{1+(x-t)t}.}$$

2. Find the unique weak solution of

$$\begin{cases} u_t + u^2 u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

which satisfies the Oleinik entropy condition.

Answer: Our characteristic equations are

$$\begin{aligned} \frac{dt}{ds} &= 1 \qquad t(r,0) = 0 \\ \frac{dx}{ds} &= z^2 \qquad x(r,0) = r \\ \frac{dz}{ds} &= 0 \qquad z(r,0) = \phi(r). \end{aligned}$$

The solution of the characteristic equations is

$$\begin{aligned} t &= s \\ z &= \phi(r) \\ x &= [\phi(r)]^2 t + r. \end{aligned}$$

For $r < 0$, $\phi(r) = 0$ implies $x = r$ and u is constant along these projected characteristics.

For $r > 0$, $\phi(r) = 1$ implies $x = t + r$ and u is constant along these projected characteristics.

Therefore, we have a wedge in which u is not defined. We fill in this wedge with a rarefaction wave. Therefore, our solution is given by

$$u(x,t) = \begin{cases} 0 & x < 0 \\ G(x/t) & 0 < x < t \\ 1 & x > t \end{cases}$$

where $G(y) = (f')^{-1}(y)$ for $y > 0$. We have $f(y) = \frac{y^3}{3}$ which implies $f'(y) = y^2$, and, therefore, $(f')^{-1}(y) = \sqrt{y}$. We conclude that our solution is

$$\boxed{u(x,t) = \begin{cases} 0 & x < 0 \\ \sqrt{x/t} & 0 < x < t \\ 1 & x > t \end{cases}}$$

3. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi\}$. Find all eigenvalues and eigenfunctions for the following eigenvalue problem

$$\begin{cases} -\Delta X(x,y) = \lambda X(x,y) & (x,y) \in \Omega \\ X(0,y) = X_x(\pi,y) = 0, & 0 < y < \pi \\ X(x,0) = X_y(x,\pi) = 0, & 0 < x < \pi. \end{cases}$$

Answer:

Use separation of variables. Let $X(x,y) = S(x)Y(y)$. This implies

$$-S''Y - SY'' = \lambda SY.$$

Dividing by SY, we have

$$-\frac{S''}{S} - \frac{Y''}{Y} = \lambda,$$

which implies

$$-\frac{S''}{S} = \frac{Y''}{Y} + \lambda = \mu.$$

First, we consider the eigenvalue problem

$$\begin{cases} -S'' = \mu S \\ S(0) = 0 = S'(\pi). \end{cases}$$

We look for positive eigenvalues $\mu = \beta^2 > 0$. This implies

$$S(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$S(0) = 0 \implies A = 0.$$

The boundary condition

$$S'(\pi) = 0 \implies B\beta\cos(\beta\pi) = 0 \implies \beta\pi = \left(n + \frac{1}{2}\right)\pi.$$

Therefore,

$$\mu = \beta^2 = \left(n + \frac{1}{2}\right)^2.$$

We look to see if zero is an eigenvalue. If $\mu = 0$, then

$$S(x) = A + Bx.$$

The boundary condition

$$S(0) = 0 \implies A = 0.$$

The boundary condition

$$S'(\pi) = 0 \implies B = 0.$$

Therefore, $S(x) \equiv 0$, but the zero function is not an eigenfunction. Therefore, zero is not an eigenvalue.

We look for negative eigenvalues, $\mu = -\gamma^2 < 0$. This implies

$$S(x) = A\cosh(\gamma x) + B\sinh(\gamma x).$$

The boundary condition

$$S(0) = 0 \implies A = 0.$$

The boundary condition

$$S'(\pi) = 0 \implies B\gamma\cosh(\gamma\pi) = 0 \implies B = 0.$$

Again, this implies $S(x) \equiv 0$, but the zero function is not an eigenfunction. Therefore, there are no negative eigenvalues.

Therefore, we have a sequence, $S_n(x) = \sin\left(\left(n + \frac{1}{2}\right)x\right)$.

Now, Y satisfies the equation

$$-\frac{Y''}{Y} = \lambda - \mu.$$

From the same analysis as above, we conclude that $\lambda - \mu = \left(m + \frac{1}{2}\right)^2$ and $Y_m(y) = \sin\left(\left(m + \frac{1}{2}\right) y\right)$.

Therefore, our eigenvalues and eigenfunctions are

$$\boxed{X_{mn}(x,y) = \sin\left(\left(n + \frac{1}{2}\right) x\right) \sin\left(\left(m + \frac{1}{2}\right) y\right) \qquad \lambda_{mn} = \left(m + \frac{1}{2}\right)^2 + \left(n + \frac{1}{2}\right)^2.}$$

4. Find the solution of the following initial-value problem for an inhomogeneous wave equation in $\mathbb{R}^3$,

$$\begin{cases} u_{tt} - \Delta u = |x|^2 & x \in \mathbb{R}^3 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

You should explicitly calculate any integrals involved in the solution.

Answer: The solution is given by

$$u(x,t) = \int_0^t \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} |y|^2 \, dS(y).$$

Now

$$\begin{aligned} \int_{\partial B(x,t-s)} |y|^2 \, dS(y) &= 4\pi(t-s)^2 \fint_{\partial B(x,t-s)} |y|^2 \, dS(y) \\ &= 4\pi(t-s)^2 \fint_{\partial B(0,1)} |x + (t-s)z|^2 \, dS(z) \\ &= 4\pi(t-s)^2 \fint_{\partial B(0,1)} (|x|^2 + 2(t-s)x \cdot z + (t-s)^2|z|^2) \, dS(z) \\ &= 4\pi(t-s)^2[|x|^2 + (t-s)^2]. \end{aligned}$$

Therefore,

$$\begin{aligned} u(x,t) &= \int_0^t \frac{1}{4\pi(t-s)} \left[4\pi(t-s)^2[|x|^2 + (t-s)^2]\right] \, ds \\ &= \int_0^t \left[|x|^2(t-s) + (t-s)^3\right] \, ds \\ &= |x|^2 \frac{(t-s)^2}{-2}\Big|_{s=0}^{s=t} + \frac{(t-s)^4}{-4}\Big|_{s=0}^{s=t} \\ &= \frac{|x|^2 t^2}{2} + \frac{t^4}{4}. \end{aligned}$$

Therefore,

$$\boxed{u(x,t) = \frac{|x|^2 t^2}{2} + \frac{t^4}{4}.}$$

5. Let u be the solution of the wave equation in three dimensions,

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^3 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

where the initial data ϕ, ψ is supported in the ball of radius R about the origin. Let x_0 be a point in $\mathbb{R}^3$ with $|x_0| > R$.

(a) Find the largest time T_1 for which we can guarantee that $u(x_0, t)$ must be zero for all $0 \leq t < T_1$.

Answer: We know the value of the solution at the point (x, t) depends on the values of the initial data on the boundary of the ball of radius ct about x. Therefore, for $t > 0$, $|x_0| >$, our solution $u = 0$ if $|x_0| - ct > R$, because this ball will not intersect the ball of radius R about the origin (the support of the initial data). Therefore,

$$\boxed{T_1 = \frac{|x_0| - R}{c}.}$$

(b) Find the smallest time T_2 such that we can guarantee that $u(x_0, t)$ must be zero for all $t > T_2$.

Answer: We also know that the solution will be zero if the ball of radius ct about the point x_0 encloses the ball of radius R; that is, $ct - |x_0| > R$. Therefore,

$$\boxed{T_2 = \frac{|x_0| + R}{c}.}$$

6. Consider the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X(0) - a_1 X'(0) = 0 \\ X(l) + a_2 X'(l) = 0 \end{cases}$$

where $a_1, a_2 \geq 0$.

(a) Prove all eigenvalues are positive.

Answer: We check if $\lambda = 0$ is an eigenvalue. In this case,

$$X(x) = A + Bx.$$

Now the boundary condition

$$X(0) - a_1 X'(0) = 0 \implies A - a_1 B = 0.$$

The boundary condition

$$X(l) + a_2 X'(l) = 0 \implies A + Bl + a_2 B = 0.$$

Combining these two equations, we have

$$a_1B + Bl + a_2B = 0 \implies B[a_1 + l + a_2] = 0,$$

which implies either $B = 0$ or $a_1 + l + a_2 = 0$. But, $a_1, a_2 \geq 0$ and $l > 0$. Therefore, we must have $B = 0$. But, $A = a_1B$ implies $A = 0$, and, therefore, $X(x) \equiv 0$. But, the zero function is not an eigenfunction. Therefore, zero is not an eigenvalue.

Next, we check if we have any negative eigenvalues $\lambda = -\gamma^2 < 0$. In this case,

$$X(x) = A\cosh(\gamma x) + B\sinh(\gamma x).$$

The boundary condition

$$X(0) - a_1X'(0) = 0 \implies A = a_1B\gamma.$$

The boundary condition

$$X(l) + a_2X'(l) = 0 \implies A\cosh(\gamma l) + B\sinh(\gamma l) + a_2A\gamma\sinh(\gamma l) + a_2B\gamma\cosh(\gamma l).$$

Combining the two equations, we have

$$B[(a_1\gamma + a_2\gamma)\cosh(\gamma l) + (1 + a_1a_2\gamma^2)\sinh(\gamma l)] = 0$$

which implies either $B = 0$ or $(a_1\gamma + a_2\gamma)\cosh(\gamma l) + (1 + a_1a_2\gamma^2)\sinh(\gamma l) = 0$. But, for $\gamma, l > 0$, $a_1, a_2 \geq 0$, $(1 + a_1a_2\gamma^2)\sinh(\gamma l) > 0$, and, therefore, the second equation cannot equal zero. Therefore, $B = 0$. But, $A = a_1B\gamma$ implies $A = 0$. Therefore, $X(x) \equiv 0$. But, again by definition, the zero function is not an eigenfunction. Therefore, there are no negative eigenvalues.

(b) Show there is an infinite sequence of positive eigenvalues λ_n such that $\lambda_n \to +\infty$ as $n \to +\infty$. (You may need to use an implicit formula to define your eigenvalues.)

Answer:

We look for positive eigenvalues $\lambda = \beta^2 > 0$. In this case,

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

The boundary condition

$$X(0) - a_1X'(0) = 0 \implies A - a_1B\beta = 0 \implies A = a_1B\beta.$$

The boundary condition

$$X(l) + a_2X'(l) = 0 \implies A\cos(\beta l) + B\sin(\beta l) + a_2[-A\beta\sin(\beta l) + B\beta\cos(\beta l)] = 0.$$

Combining these two equations, we have

$$a_1B\beta\cos(\beta l) + B\sin(\beta l) - a_2a_1B\beta^2\sin(\beta l) + a_2B\beta\cos(\beta l) = 0,$$

which implies

$$B[(a_1\beta + a_2\beta)\cos(\beta l) + (1 - a_1 a_2\beta^2)\sin(\beta l)] = 0.$$

We don't want $B = 0$ because this would imply $A = 0$ and $X(x) \equiv 0$ is not an eigenfunction. Therefore, we need

$$[(a_1\beta + a_2\beta)\cos(\beta l) + (1 - a_1 a_2\beta^2)\sin(\beta l)] = 0.$$

In particular, β is given implicitly by

$$\tan(\beta l) = \frac{(a_1 + a_2)\beta}{a_1 a_2 \beta^2 - 1}$$

and the eigenvalues are $\lambda_n = \beta_n^2$. By graphing $f(\beta) = \tan(\beta l)$ and $g(\beta) = \frac{(a_1+a_2)\beta}{a_1 a_2\beta^2 - 1}$, we see these graphs intersect an infinite number of times, and, consequently there are an infinite number of positive eigenvalues.

(c) Find the eigenfunction X_n associated with each eigenvalue λ_n.

Answer:

$$\boxed{X_n(x) = a_1\beta_n \cos(\beta_n x) + \sin(\beta_n x).}$$

where

$$\tan(\beta_n l) = \frac{(a_1 + a_2)\beta_n}{a_1 a_2 \beta_n^2 - 1}.$$

7. Let λ_n, X_n be the eigenvalues and eigenfunctions of the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X(0) - a_1 X'(0) = 0 \\ X(l) + a_2 X'(l) = 0 \end{cases}$$

where $a_1, a_2 \geq 0$. By the previous problem, we know each $\lambda_n > 0$. Use separation of variables to solve the following initial-value problem,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < l \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) - a_1 u_x(0,t) = 0 \\ u(l,t) + a_2 u_x(l,t) = 0 \end{cases}$$

where $a_1, a_2 \geq 0$. You do **not** need to evaluate any integrals. You should express your solution in terms of ϕ, ψ, f, λ_n, and X_n.

Answer: First, we look for a solution of the homogeneous equation. Using separation of variables, we look for a solution of the form $u(x,t) = X(x)T(t)$. This leads us to the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X(0) - a_1 X'(0) = 0 \\ X(l) + a_2 X'(l) = 0. \end{cases}$$

By the previous problem, there is an infinite sequence of positive eigenvalues λ_n with corresponding eigenfunctions $X_n(x)$. Therefore, the solution of the homogeneous equation is given by

$$u_h(x,t) = \sum_{n=1}^{\infty} [A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)] X_n(x)$$

where

$$\begin{aligned} A_n &= \frac{\langle \phi, X_n \rangle}{\langle X_n, X_n \rangle} \\ c\sqrt{\lambda_n} B_n &= \frac{\langle \psi, X_n \rangle}{\langle X_n, X_n \rangle}, \end{aligned}$$

where

$$\langle f, g \rangle = \int_0^l f(x) g(x)\, dx.$$

Therefore, using Duhamel's principle, the solution of the inhomogeneous equation is given by

$$\boxed{\begin{aligned} u(x,t) = &\sum_{n=1}^{\infty} [A_n \cos(c\sqrt{\lambda_n} t) + B_n \sin(c\sqrt{\lambda_n} t)] X_n(x) \\ &+ \int_0^t \sum_{n=1}^{\infty} C_n(s) \sin(c\sqrt{\lambda_n}(t-s)) X_n(x)\, ds \end{aligned}}.$$

where A_n and B_n are defined above, and

$$c\sqrt{\lambda_n} C_n(s) = \frac{\langle f(s), X_n \rangle}{\langle X_n, X_n \rangle}.$$

8. Consider the initial-value problem for the telegrapher's equation,

$$(*) \begin{cases} u_{tt} - c^2 u_{xx} + a u_t = 0 \qquad a > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

Define the energy function associated with this equation as

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} [u_t^2 + c^2 u_x^2]\, dx.$$

(a) Prove that for solutions u of (*) which have compact support, $E(t)$ is a non-increasing function of t.

Answer: Using the assumption that u has compact support, and that u satisfies (*), we have

$$\begin{aligned}
E'(t) &= \int_{-\infty}^{\infty} [u_t u_{tt} + c^2 u_x u_{xt}]\, dx \\
&= \int_{-\infty}^{\infty} [u_t u_{tt} - c^2 u_{xx} u_t]\, dx + c^2 u_x u_t \Big|_{x\to-\infty}^{x\to+\infty} \\
&= \int_{-\infty}^{\infty} u_t [u_{tt} - c^2 u_{xx}]\, dx \\
&= -\int_{-\infty}^{\infty} a u_t^2 \, dx \\
&\le 0,
\end{aligned}$$

as claimed.

(b) Use the energy function to prove there exists at most one smooth solution of the initial-value problem (*) which has compact support.

Answer: Suppose there exist two solutions u and v. Let $w = u - v$. Therefore, w is a solution of

$$\begin{cases} w_{tt} - c^2 w_{xx} + a w_t = 0 \\ w(x,0) = 0 \\ w_t(x,t) = 0. \end{cases}$$

Therefore,

$$E_w(t) = \frac{1}{2} \int_{-\infty}^{\infty} [w_t^2 + c^2 w_x^2]\, dx$$

implies

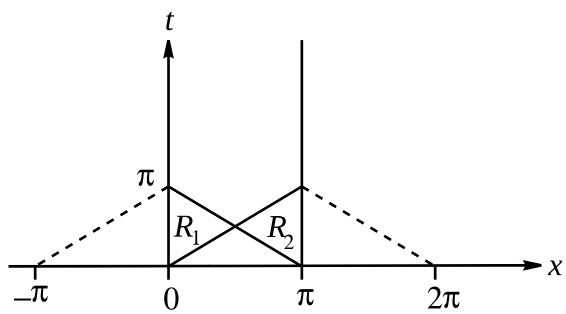
$$E_w(0) = \frac{1}{2} \int_{-\infty}^{\infty} [w_t^2(x,0) + c^2 w_x^2(x,0)]\, dx = 0.$$

By part (a), we know that $E_w'(t) \le 0$. Therefore, $E_w(t) \le 0$ for all $t \ge 0$. But, by definition of E_w, we see that $E_w(t) \ge 0$. Therefore, $E_w(t) \equiv 0$ for all $t \ge 0$. Therefore, using the fact that w is a smooth solution, we conclude that $w_t(x,t) = 0 = w_x(x,t)$ for all $x \in \mathbb{R}$. Therefore, $w(x,t) \equiv C$ for some constant C. But, by assumption, $w(x,0) = 0$. Therefore, $C = 0$. Consequently, we conclude that $w(x,t) \equiv 0$, and, therefore, $u(x,t) = v(x,t)$.

9. Let u be the solution of the wave equation on the interval $[0,l]$ with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi \\ u(x,0) = x \\ u_t(x,0) = \sin(x) \\ u(0,t) = 0 = u(l,t). \end{cases}$$

Use the method of reflection to find the values of u in regions R_1 and R_2 shown below.



Answer:

We extend our initial data to be odd functions with respect to $x = 0$ and $x = l$. That is, introduce new functions ϕ_{ext} and ψ_{ext} such that

$$\phi_{ext}(x) = x - 2n\pi \qquad -\pi + 2n\pi < x < \pi + 2n\pi$$

for $n \in \mathbb{Z}$. Now $\psi(x) = \sin(x)$ is already 2π-periodic. Therefore, we let $\psi_{ext}(x) = \sin(x)$ for all x. Then we find a solution of the equation on $(0, \pi)$ by looking for a solution to

$$\begin{cases} u_{tt} - u_{xx} = 0 & -\infty < x < \infty \\ u(x,0) = \phi_{ext}(x) \\ u_t(x,0) = \psi_{ext}. \end{cases}$$

We know the solution of this equation on the whole line is given by d'Alembert's formula,

$$u(x,t) = \frac{1}{2}[\phi_{ext}(x+t) + \phi_{ext}(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \psi_{ext}(y)\,dy.$$

For $(x,t) \in R_1$, $-\pi < x - t < 0$. Therefore, $\phi_{ext}(x-t) = x - t$ and $\psi_{ext}(x-t) = \sin(x)$. Therefore, for $(x,t) \in R_1$, our solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}[(x+t) + (x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \sin(y)\,dy \\ &= x - \frac{1}{2}\cos(x+t) + \frac{1}{2}\cos(x-t). \end{aligned}$$

For $(x,t) \in R_2$, $\pi < x + t < 2\pi$. Therefore, $\phi_{ext}(x+t) = x + t - 2\pi$. Therefore,

$$\begin{aligned} u(x,t) &= \frac{1}{2}[(x+t-2\pi) + (x-t)] + \frac{1}{2}\int_{x-t}^{x+t} \sin(y)\,dy \\ &= x - \pi - \frac{1}{2}\cos(x+t) + \frac{1}{2}\cos(x-t). \end{aligned}$$

10. Answer true or false to the following questions. No explanation is necessary.

(a) There exists a unique solution of the following initial-value problem,

$$\begin{cases} u_t + xu_x = 0 & x \in \mathbb{R} \\ u(x,0) = \phi(x) \end{cases}$$

for all smooth functions ϕ.

Answer: True. (The curve $\Gamma = (r, 0)$ is noncharacteristic. Therefore, there exists a unique solution.)

(b) All eigenvalues of $-\partial_x^2$ are positive if the boundary conditions are symmetric.

Answer: False. (For example, Neumann boundary conditions are symmetric, but zero is an eigenvalue. Robin boundary conditions are symmetric, but some Robin boundary conditions have negative eigenvalues.)

(c) The interval of dependence for the solution of the inhomogeneous wave equation on the half-line with Neumann boundary conditions is $[0, x + ct]$ for the point (x, t) such that $x - ct < 0$.

Answer: True. (Consider

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < +\infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u_x(0,t) = 0. \end{cases}$$

For a point (x, t) such that $x - ct < 0$, the solution is given by

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy \\ &= \frac{1}{2}[\phi(x+ct) + \phi(ct-x)] + \frac{1}{2c}\int_0^{ct-x} \psi(y)\,dy + \frac{1}{2c}\int_0^{x+ct} \psi(y)\,dy. \end{aligned}$$

Therefore, for the homogeneous equation with Neumann boundary conditions, the value of the solution at the point (x, t) depends on the values of the initial data in the interval $(0, x+ct)$. Similarly, you can use Duhamel's principle to show that the same is true for the inhomogeneous equation.)

(d) If $f \in C^1([-l, l])$, then the full Fourier series of f converges to f pointwise in $(-l, l)$.

Answer: True. (See the theorem from the lecture notes on pointwise convergence of the full Fourier series.)

(e) Consider the wave equation

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

where n is odd. The value of the solution u at the point (x_0, t_0) depends at most on the value of the initial data on $\partial B(x_0, ct_0)$, where $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$.

Answer: False. (This is true if n is odd *and* $n \geq 3$. But, for $n = 1$, we see by d'Alembert's formula that the solution at the point (x,t) depends on the value of the initial data in the entire ball.)

(f) Every linear, constant-coefficient, second-order equation of the form

$$\sum_{i,j} a_{ij} u_{x_i x_j} = 0 \qquad x \in \mathbb{R}^n$$

can be reduced to one of the following forms through a change of variables,

$$u_{y_1 y_1} - \sum_{i=2}^{n} u_{y_i y_i} = 0 \text{ (hyperbolic)}$$
$$\sum_{i=1}^{n} u_{y_i y_i} = 0 \text{ (elliptic)}$$
$$\sum_{i=2}^{n} u_{y_i y_i} = 0 \text{ (parabolic)}$$

Answer: False. (We classify second-order, linear equations based on the eigenvalues of the symmetric, coefficient matrix $A = (a_{ij})$. Under certain conditions on the eigenvalues, the equation will be classified as elliptic, hyperbolic, or parabolic. However, not all equations will fall into one of these categories.)

(g) Consider the following initial-value problem,

$$\begin{cases} u_t + [f(u)]_x = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

where $\phi(x)$ is piecewise constant and f is uniformly convex. If there exists a point $x_1 > x_2$ such that $\phi(x_1) < \phi(x_2)$, then the unique weak, admissible solution u will have a shock curve.

Answer: True. (If the initial data is piecewise constant such that $\phi(x) = u^-$ for $x < x_0$ and $\phi(x) = u^+$ for $x > x_0$, where $u^- > u^+$, then the unique, weak admissible solution must have a shock curve. See the theorem on the Riemann problem in the notes, or in Evans.)

(h) Suppose f is a uniformly convex function. Then the entropy condition and the Oleinik entropy condition are equivalent.

Answer: True. (If f is uniformly convex, then

$$f'(u^-) > \sigma > f'(u^+) \iff \frac{f(u^-) - f(u)}{f(u^+) - f(u)} \geq \frac{f(u^-) - f(u^+)}{u^- - u^+} \qquad \forall u \in (u^+, u^-),$$

where $\sigma \equiv (f(u^-) - f(u^+))/(u^- - u^+)$.

(i) For first-order semilinear equations, projected characteristics do not intersect, and, therefore, solutions (and their derivatives) will not blow-up.

Answer: False. (For example, see problem 1 on the exam. This is a semilinear equation. The projected characteristics do not intersect, but the solution will blow-up in finite time.)

(j) The function

$$u(x,t) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

is a weak solution of

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases}$$

Answer: False. (The Rankine-Hugoniot jump condition is not satisfied.)