Math 220A Practice Final Exam 2 Fall 2002

1. Solve the following initial-value problem,

$$\begin{cases} u_t + u_x + u^2 = 0 \\ u(x,0) = x. \end{cases}$$

2. Find the unique weak solution of

$$\begin{cases} u_t + u^2 u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

which satisfies the Oleinik entropy condition.

3. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi\}$. Find all eigenvalues and eigenfunctions for the following eigenvalue problem

$$\begin{cases} -\Delta X(x,y) = \lambda X(x,y) & (x,y) \in \Omega \\ X(0,y) = X_x(\pi,y) = 0, & 0 < y < \pi \\ X(x,0) = X_y(x,\pi) = 0, & 0 < x < \pi. \end{cases}$$

4. Find the solution of the following initial-value problem for an inhomogeneous wave equation in $\mathbb{R}^3$,

$$\begin{cases} u_{tt} - \Delta u = |x|^2 & x \in \mathbb{R}^3 \\ u(x,0) = 0 \\ u_t(x,0) = 0. \end{cases}$$

You should explicitly calculate any integrals involved in the solution.

5. Let u be the solution of the wave equation in three dimensions,

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^3 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

where the initial data ϕ, ψ is supported in the ball of radius R about the origin. Let x_0 be a point in $\mathbb{R}^3$ with $|x_0| > R$.

(a) Find the largest time T_1 for which we can guarantee that $u(x_0,t)$ must be zero for all $0 \le t < T_1$.

(b) Find the smallest time T_2 such that we can guarantee that $u(x_0,t)$ must be zero for all $t > T_2$.

6. Consider the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X(0) - a_1 X'(0) = 0 \\ X(l) + a_2 X'(l) = 0 \end{cases}$$

where $a_1, a_2 \geq 0$.

(a) Prove all eigenvalues are positive.

(b) Show there is an infinite sequence of positive eigenvalues λ_n such that $\lambda_n \to +\infty$ as $n \to +\infty$. (You may need to use an implicit formula to define your eigenvalues.)

(c) Find the eigenfunction X_n associated with each eigenvalue λ_n.

7. Let λ_n, X_n be the eigenvalues and eigenfunctions of the eigenvalue problem

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X(0) - a_1 X'(0) = 0 \\ X(l) + a_2 X'(l) = 0 \end{cases}$$

where $a_1, a_2 \geq 0$. By the previous problem, we know each $\lambda_n > 0$. Use separation of variables to solve the following initial-value problem,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < l \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) - a_1 u_x(0,t) = 0 \\ u(l,t) + a_2 u_x(l,t) = 0 \end{cases}$$

where $a_1, a_2 \geq 0$. You do **not** need to evaluate any integrals. You should express your solution in terms of ϕ, ψ, f, λ_n, and X_n.

8. Consider the initial-value problem for the telegrapher's equation,

$$(*) \begin{cases} u_{tt} - c^2 u_{xx} + a u_t = 0 & a > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

Define the energy function associated with this equation as

$$E(t) = \frac{1}{2} \int_{-\infty}^{\infty} [u_t^2 + c^2 u_x^2] \, dx.$$

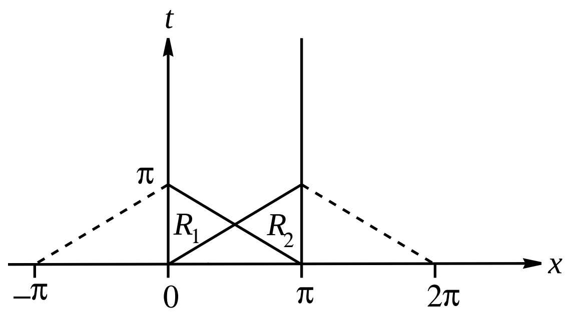
(a) Prove that for solutions u of (*) which have compact support, $E(t)$ is a non-increasing function of t.

(b) Use the energy function to prove there exists at most one smooth solution of the initial-value problem (*) which has compact support.

9. Let u be the solution of the wave equation on the interval $[0, l]$ with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi \\ u(x,0) = x \\ u_t(x,0) = \sin(x) \\ u(0,t) = 0 = u(l,t). \end{cases}$$

Use the method of reflection to find the values of u in regions R_1 and R_2 shown below.



10. Answer true or false to the following questions. No explanation is necessary.

(a) There exists a unique solution of the following initial-value problem,

$$\begin{cases} u_t + x u_x = 0 & x \in \mathbb{R} \\ u(x,0) = \phi(x) \end{cases}$$

for all smooth functions ϕ.

(b) All eigenvalues of $-\partial_x^2$ are positive if the boundary conditions are symmetric.

(c) The interval of dependence for the solution of the inhomogeneous wave equation on the half-line with Neumann boundary conditions is $[0, x + ct]$ for the point (x, t) such that $x - ct < 0$.

(d) If $f \in C^1([-l, l])$, then the full Fourier series of f converges to f pointwise in $(-l, l)$.

(e) Consider the wave equation

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

where n is odd. The value of the solution u at the point (x_0, t_0) depends at most on the value of the initial data on $\partial B(x_0, ct_0)$, where $B(x, r) = \{y \in \mathbb{R}^n : |y - x| < r\}$.

(f) Every linear, constant-coefficient, second-order equation of the form

$$\sum_{i,j} a_{ij} u_{x_i x_j} = 0 \qquad x \in \mathbb{R}^n$$

can be reduced to one of the following forms through a change of variables,

$$u_{y_1y_1} - \sum_{i=2}^{n} u_{y_iy_i} = 0 \text{ (hyperbolic)}$$
$$\sum_{i=1}^{n} u_{y_iy_i} = 0 \text{ (elliptic)}$$
$$\sum_{i=2}^{n} u_{y_iy_i} = 0 \text{ (parabolic)}$$

(g) Consider the following initial-value problem,

$$\begin{cases} u_t + [f(u)]_x = 0 & x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

where $\phi(x)$ is piecewise constant and f is uniformly convex. If there exists a point $x_1 > x_2$ such that $\phi(x_1) < \phi(x_2)$, then the unique weak, admissible solution u will have a shock curve.

(h) Suppose f is a uniformly convex function. Then the entropy condition and the Oleinik entropy condition are equivalent.

(i) For first-order semilinear equations, projected characteristics do not intersect, and, therefore, solutions (and their derivatives) will not blow-up.

(j) The function

$$u(x,t) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

is a weak solution of

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0. \end{cases}$$