Math 220A
Practice Final Exam I - Fall 2002

1. (a) Suppose $S(t)$ is the solution operator associated with the homogeneous equation

$$(*) \begin{cases} u_t + au_x = 0 \\ u(x,0) = \phi(x). \end{cases}$$

In particular, assume the solution of (*) is given by $u(x,t) = S(t)\phi(x)$. Show that $v(x,t) = S(t)\phi(x) + \int_0^t S(t-s)f(x,s)\,ds$ solves the inhomogeneous problem

$$\begin{cases} u_t + au_x = f(x,t) \\ u(x,0) = 0. \end{cases}$$

(b) Find the solution operator $S(t)$ for (*).

(c) Find a solution of the inhomogeneous initial-value problem

$$\begin{cases} u_t + au_x = f(x,t) \\ u(x,0) = \phi(x). \end{cases}$$

2. (a) Solve the following initial-value problem.

$$\begin{cases} u_x^2 u_t - 1 = 0 \\ u(x,0) = x. \end{cases}$$

(b) Consider the initial-value problem

$$\begin{cases} u_t + u_x = x \\ u(x,x) = 1. \end{cases}$$

Explain why there is no solution to this problem.

3. (a) Find the general solution of

$$u_{tt} + 2u_{xt} - 3u_{xx} = 0.$$

(b) Find the solution of the initial-value problem,

$$\begin{cases} u_{tt} + 2u_{xt} - 3u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

4. Consider the initial-value problem

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} a & x \le 0 \\ a(1-x) & 0 < x < 1 \\ 0 & x \ge 1 \end{cases}$$

where $a > 0$. Find the unique, weak solution which satifies the entropy condition.

5. Consider the initial-value problem

$$\begin{cases} u_{tt} + 2u_{xt} - 3u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

(a) Use energy methods to prove the value of the solution u at the point (x_0, t_0) depends at most on the values of the initial data in the interval $(x_0 - 3t_0, x_0 + t_0)$.

(b) Use energy methods to prove uniqueness of solutions to this initial-value problem if the initial data has compact support.

6. Consider the following eigenvalue problem.

$$\begin{cases} y'' + \lambda y = 0, & 0 < x < l \\ y'(0) + y(0) = 0 \\ y(l) = 0. \end{cases}$$

(a) Show the boundary conditions are symmetric.

(b) State the definition of orthogonality of functions on $[0, l]$.

(c) Use the fact that the boundary conditions are symmetric to prove all eigenfunctions of this operator must be orthogonal.

(d) Find all *positive* eigenvalues and their corresponding eigenfunctions. (Note: You may not be able to find an explicit formula for these eigenvalues.) Show graphically that there are an infinite number of positive eigenvalues $\{\lambda_n\}$ such that $\lambda_n \to +\infty$.

7. Consider the following initial/boundary value problem,

$$\begin{cases} u_{tt} - 4u_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = 0 & 0 < x < l \\ u_t(x,0) = 0 & 0 < x < l \\ u(0,t) = \sin t \\ u(l,t) = 1. \end{cases}$$

Define a function $\mathcal{U}(x,t)$ such that by letting $v(x,t) = u(x,t) - \mathcal{U}(x,t)$, then $v(x,t)$ will satisfy

$$\begin{cases} v_{tt} - 4v_{xx} = f(x,t) & 0 < x < l, t > 0 \\ v(x,0) = \phi(x) & 0 < x < l \\ v_t(x,0) = \psi(x) & 0 < x < l \\ v(0,t) = 0 = v(l,t) & t > 0 \end{cases}$$

for some functions $f(x,t), \phi(x)$ and $\psi(x)$, thus, reducing the problem with inhomogeneous boundary data to an inhomogeneous problem with Dirichlet boundary data. You do **not** need to solve the new inhomogeneous problem.

8. Consider the initial-value problem for the wave equation in n dimensions,

$$\begin{cases} u_{tt} - \Delta u = 0 & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

   (a) If the initial data is supported in the annular region $\{a < |x| < b\}$, find where the solution is definitely zero in

      i. $\mathbb{R}^2$

      ii. $\mathbb{R}^3$.

   (b) Find the value of the solution u of the initial-value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & x \in \mathbb{R}^3, t \geq 0 \\ u(x,0) = 0 \\ u_t(x,0) = \psi(x) \end{cases}$$

   where

$$\psi(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

   at a point (x,t) such that $|x| + t < a$.

9. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 < x < \pi, 0 < y < \pi\}$. Solve the following initial/boundary value problem.

$$\begin{cases} u_{tt} = u_{xx} + u_{yy} + 1 & (x,y) \in \Omega, t > 0 \\ u(x,y,0) = \sin(x)\sin(2y) \\ u_t(x,y,0) = 0 \\ u(x,y,t) = 0 & (x,y) \in \partial\Omega. \end{cases}$$

10. Use Green's Theorem to show that the value of the solution u at the point $(0,t_0)$ of the wave equation on the half-line with Neumann boundary conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u_x(0,t) = 0 \end{cases}$$

is given by

$$u(0,t_0) = \phi(ct_0) + \frac{1}{c}\int_0^{ct_0} \psi(y)\,dy + \frac{1}{c}\iint_\Delta f(y,s)\,dy\,ds$$

where Δ is the triangle in the xt-plane bounded by the lines $x = 0$, $t = 0$ and $x = c(t_0 - t)$.