

1. (10 points) Solve

$$\begin{cases} u_t^2 - c^2 u_x^2 = 0 \\ u(0, t) = \phi(t) \end{cases}$$

**Answer:** Let  $F(x, t, z, p, q) = q^2 - c^2 p^2$ . Using the method of characteristics, we define the system of characteristic ODEs as follows,

$$\begin{aligned} \frac{dx}{ds} = F_p = -2c^2 p & & x(r, 0) = 0 \\ \frac{dt}{ds} = F_q = 2q & & t(r, 0) = r \\ \frac{dz}{ds} = pF_p + qF_q = -2c^2 p^2 + 2q^2 & & z(r, 0) = \phi(r) \\ \frac{dp}{ds} = -F_x - pF_z = 0 & & p(r, 0) = \psi_1(r) \\ \frac{dq}{ds} = -F_t - qF_z = 0 & & q(r, 0) = \psi_2(r) \end{aligned}$$

where we will determine  $\psi_1, \psi_2$  below. Using the fact that we want

$$u_r(x(r, s), t(r, s)) = u_x(x(r, s), t(r, s))x_r(r, s) + u_t(x(r, s), t(r, s))t_r(r, s)$$

we want

$$\phi'(r) = \psi_1(r) \cdot 0 + \psi_2(r) \cdot 1.$$

Therefore, we see that  $\psi_2(r) = \phi'(r)$ .

Next, using the fact that we need  $F(x, t, z, p, q) = 0$ , we see that we need

$$\psi_2^2 - c^2 \psi_1^2 = 0 \implies \psi_1 = \pm \frac{\phi'}{c}.$$

Now solving our system of characteristic ODEs (taking into account our initial conditions), we have

$$\begin{aligned} \frac{dq}{ds} = 0 & & \implies q(r, s) = \psi_2(r) = \phi'(r) \\ \frac{dp}{ds} = 0 & & \implies p(r, s) = \psi_1(r) = \pm \frac{\phi'}{c} \\ \frac{dx}{ds} = -2c^2 p = \mp 2c\phi' & & \implies x(r, s) = \mp 2c\phi'(r)s \\ \frac{dt}{ds} = 2q = 2\phi' & & \implies t(r, s) = 2\phi'(r)s + r \\ \frac{dz}{ds} = -2c^2 p^2 + 2q^2 = -2(\phi')^2 + 2(\phi')^2 = 0 & & \implies z(r, s) = \phi(r) \end{aligned}$$

Now  $\phi'(r)s = \frac{x}{\mp 2c}$  implies  $t = \frac{\mp x}{c} + r$ . Therefore, we conclude our solution is given by

$$\boxed{u(x, t) = \phi\left(t \pm \frac{x}{c}\right)}.$$

2. (12 points) Find the unique, weak solution of

$$\begin{cases} u_t + [u^3]_x = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 2 & x > 0 \\ -2 & x < 0 \end{cases}$$

which satisfies the Oleinik entropy condition. Simplify your answer as much as possible.

**Answer:** The characteristic equations are given by

$$\begin{aligned} \frac{dx}{ds} &= 3z^2 & x(r, 0) &= r \\ \frac{dt}{ds} &= 1 & t(r, 0) &= 0 \\ \frac{dz}{ds} &= 0 & z(r, 0) &= \phi(r). \end{aligned}$$

Solving this system, we have

$$\begin{aligned} x(r, s) &= 3\phi^2 s + r \\ t(r, s) &= s \\ z(r, s) &= \phi(r). \end{aligned}$$

Therefore, our projected characteristics are given by  $x = 3\phi^2(r)t + r$ . By our initial conditions, we see that  $u(x, t) = -2$  to the left of  $x = 12t$  and  $u(x, t) = 2$  to the right of  $x = 12t$ . Now we see that the solution would have a jump across this curve. We notice, however, that the RH jump condition would not be satisfied across  $\xi(t) = 12t$ , because

$$\frac{[f(u)]}{[u]} = \frac{u_-^3 - u_+^3}{u_- - u_+} = \frac{-8 - 8}{-2 - 2} = 4 \neq 12 = \xi'(t).$$

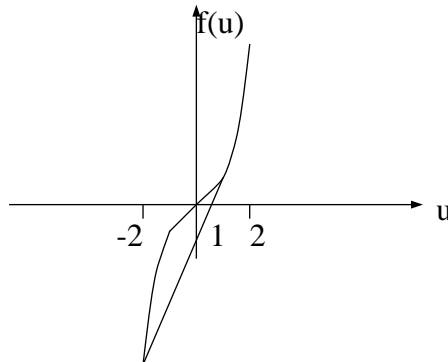
We look for a weak solution (in particular, satisfying the RH jump condition) and the Oleinik entropy condition. We use the rubberband method.

In order to satisfy the Oleinik entropy condition, we will put in a chord connecting  $u_- = -2$  and  $u_2$  such that

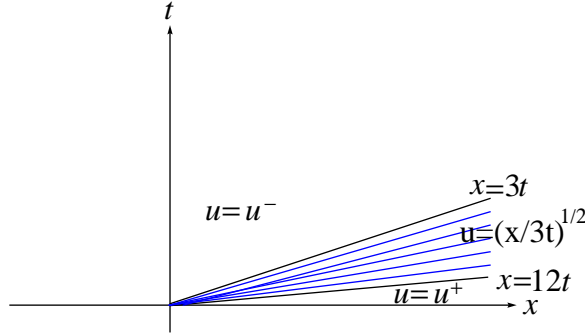
$$\frac{f(u_-) - f(u_2)}{u_- - u_2} = f'(u_2)$$

In this case, the jump condition will be satisfied, and the Oleinik entropy condition will be satisfied. Then, we notice that  $f'$  is invertible for  $u$  between  $u_2$  and  $u_+$ . Therefore, we will put in a rarefaction wave to go from  $u_2$  to  $u_+$ . Now solving for  $u_2$  we see that we need

$$3u_2^2 = \frac{u_2^3 + 8}{u_2 + 2} \implies u_2 = 1.$$



Therefore, the curve of discontinuity will be given by  $x = 3t$ . Then we put in a rarefaction wave between  $x = 3t$  and  $x = 12t$ . In this region, we define  $u(x, t) = (f')^{-1}(x/t) = \sqrt{x/(3t)}$ .



We conclude that our solution is given by

$$u(x, t) = \begin{cases} -2 & x < 3t \\ \sqrt{\frac{x}{3t}} & 3t < x < 12t \\ 2 & x > 12t \end{cases}$$

3. (12 points) Solve

$$\begin{cases} u_{tt} + 2u_{xt} - 3u_{xx} = f(x, t) & -\infty < x < \infty, t > 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

**Answer:** First, we will solve the homogeneous problem. Rewriting the homogeneous equation as

$$(\partial_t + 3\partial_x)(\partial_t - \partial_x)u = 0,$$

we will make a change of variables by introducing  $\xi$  and  $\eta$  such that

$$\begin{aligned} \partial_\xi &= \partial_t + 3\partial_x \\ \partial_\eta &= \partial_t - \partial_x. \end{aligned}$$

In particular, we want  $t_\xi = 1$ ,  $x_\xi = 3$ ,  $t_\eta = 1$ ,  $x_\eta = -1$ . Therefore, we take  $\xi = \frac{1}{4}(x + t)$ ,  $\eta = -\frac{1}{4}(x - 3t)$ . With this change of variables, our equation becomes

$$u_{\xi\eta} = 0 \implies u = f(\xi) + g(\eta).$$

In particular, the solution of the homogeneous equation is given by

$$u(x, t) = f(x + t) + g(x - 3t)$$

for some functions  $f, g$ . Now, we need to choose  $f$  and  $g$  such that our initial condition is satisfied. Now

$$u(x, 0) = \phi(x) \implies f + g = \phi$$

and

$$u_t(x, 0) = \psi(x) \implies f' - 3g' = \psi.$$

Solving this system of simultaneous equations, we have

$$f' = \frac{1}{4}[3\phi' + \psi]g' = \frac{1}{4}[\phi' - \psi].$$

Integrating these equations, we have

$$\begin{aligned} f(x) &= \frac{3}{4}\phi + \frac{1}{4} \int_0^x \psi(y) dy + C_1 \\ g(x) &= \frac{1}{4}\phi - \frac{1}{4} \int_0^x \psi(y) dy + C_2. \end{aligned}$$

Using the fact that  $f + g = \phi$ , we see that  $C_1 + C_2 = 0$ . Therefore, the solution of our homogeneous IVP is given by

$$\begin{aligned} u(x, t) &= f(x + t) + g(x - 3t) \\ &= \frac{1}{4}[3\phi(x + t) + \phi(x - 3t)] + \frac{1}{4} \int_{x-3t}^{x+t} \psi(y) dy. \end{aligned}$$

Therefore, by Duhamel's principle, the solution of our inhomogeneous problem is given by

$$u(x, t) = \frac{1}{4}[3\phi(x + t) + \phi(x - 3t)] + \frac{1}{4} \int_{x-3t}^{x+t} \psi(y) dy + \frac{1}{4} \int_0^t \int_{x-3(t-s)}^{x+(t+s)} f(y, s) dy ds$$

4. (12 points) Consider the following initial-value problem,

$$(*) \begin{cases} u_t + [f(u)]_x = 0 & x \in \mathbb{R}, t > 0 \\ u(x, 0) = \phi(x), \end{cases}$$

where we assume  $f$  is a smooth function, such that  $f(0) = 0$ .

(a) Suppose  $u$  is a classical solution of (\*) with compact support. Show that

$$E_u(t) = \int_{-\infty}^{\infty} u^2(x, t) dx$$

is a conserved quantity. That is, show that  $E'_u(t) = 0$ .

**Answer:**

$$\begin{aligned}
 E'_u(t) &= 2 \int_{-\infty}^{\infty} uu_t dx \\
 &= -2 \int_{-\infty}^{\infty} u[f(u)]_x dx \\
 &= +2 \int_{-\infty}^{\infty} u_x[f(u)] dx - 2u[f(u)] \Big|_{x \rightarrow -\infty}^{x \rightarrow +\infty} \\
 &= 2 \int_{-\infty}^{\infty} \left( \int_0^{u(x)} f(y) dy \right)_x dx \\
 &= 2 \int_0^{u(x)} f(y) dy \Big|_{x \rightarrow -\infty}^{x \rightarrow +\infty} \\
 &= 0.
 \end{aligned}$$

- (b) Use the fact from part (a) to prove uniqueness of classical solutions with compact support of

$$\begin{cases} u_t + au_x = g(x, t) & -\infty < x < \infty, t > 0 \\ u(x, 0) = \phi(x). \end{cases}$$

**Answer:** Suppose there are two solutions  $u$  and  $v$ . Let  $w = u - v$ . Then  $w$  is a solution of

$$\begin{cases} w_t + aw_x = 0 & -\infty < x < \infty \\ w(x, 0) = 0. \end{cases}$$

Let  $E_w(t) = \int_{-\infty}^{\infty} w^2 dx$ . By part (a), we know that  $E'_w(t) = 0$ . But, we also have  $E_w(0) = 0$  by our initial conditions. Therefore, we have  $E_w(t) \equiv 0$ , which implies that  $w^2 \equiv 0$ , or  $w \equiv 0$ , which implies that  $u \equiv v$ .

5. (12 points) Let  $\phi$  be a smooth function with compact support. Consider the initial-value problem for the damped Burger's equation,

$$\begin{cases} u_t + uu_x + u = 0, & -\infty < x < \infty, t > 0 \\ u(x, 0) = \phi(x). \end{cases}$$

- (a) Find an implicit equation for the classical solution.

**Answer:** Our characteristic equations are given by

$$\begin{aligned}
 \frac{dt}{ds} &= 1 & t(r, 0) &= 0 \\
 \frac{dx}{ds} &= z & x(r, 0) &= r \\
 \frac{dz}{ds} &= -z & z(r, 0) &= \phi(r)
 \end{aligned}$$

Solving this system, we have

$$\begin{aligned}
 t &= s \\
 x &= -\phi(r)e^{-s} + r + \phi(r) \\
 z &= \phi(r)e^{-s}
 \end{aligned}$$

Therefore, our implicit solution is given by

$$\boxed{u(x, t) = \phi(x + u(1 - e^t))e^{-t}}.$$

- (b) Show that  $|u_x|$  is bounded if  $\phi'(x) \geq -1$ . That is, show that  $|u_x| \not\rightarrow +\infty$  in finite time if  $\phi'(x) \geq -1$ .

**Answer:** From our answer to part (a), we have

$$u_x = \phi'(p)e^{-t}[1 + u_x(1 - e^t)]$$

where  $p = x + u(1 - e^t)$ . Then solving this equation for  $u_x$  we have

$$u_x[1 + \phi'(p)[1 - e^{-t}]] = \phi'(p)e^{-t} \implies u_x = \frac{\phi'(p)e^{-t}}{[1 + \phi'(p)(1 - e^{-t})]}.$$

As long as  $1 + \phi'(p)(1 - e^{-t}) > 0$ , the solution will not blow up in finite time. Notice that for  $t > 0$ ,  $1 - e^{-t} > 0$ . Therefore, if  $\phi'(p) \geq -1$ , we have

$$1 + \phi'(p)(1 - e^{-t}) \geq 1 - (1 - e^{-t}) = e^{-t} > 0.$$