Math 220A **Final Exam Solutions** **December 12, 2002**

1. (10 points) Consider the following eigenvalue problem,

$$\begin{cases} -X'' = \lambda X & 0 < x < l \\ X(0) = 0 \\ X'(l) + X(l) = 0 \end{cases}$$

   (a) Find all positive eigenvalues.
   Answer: If $\lambda = \beta^2 > 0$, then we have

$$X(x) = A\cos(\beta x) + B\sin(\beta x).$$

   The boundary condition

$$X(0) = 0 \implies A = 0.$$

   The boundary condition

$$X'(l) + X(l) = 0 \implies B\beta\cos(\beta l) + B\sin(\beta l) = 0.$$

   Since we do not want $B = 0$, as that would imply $X \equiv 0$, we need $\beta\cos(\beta l) + \sin(\beta l) = 0$. Therefore, our positive eigenvalues are given by

$$\boxed{\lambda_n = \beta_n^2 \text{ where } \beta_n = -\tan(\beta_n l).}$$

   (b) Show that zero is not an eigenvalue, and that there are no negative eigenvalues.
   Answer: If $\lambda = 0$, then we have

$$X(x) = A + Bx.$$

   The boundary condition

$$X(0) = 0 \implies A = 0.$$

   The boundary condition

$$X'(l) + X(l) = 0 \implies B + Bl = 0.$$

   Now $1 + l \neq 0$ as $l > 0$. Therefore, we conclude that $B = 0$ which implies $X(x) \equiv 0$, but the zero function is not an eigenfunction. Therefore, we have no negative eigenvalues.
   If $\lambda = -\gamma^2 < 0$, then we have

$$X(x) = A\cosh(\gamma x) + B\sinh(\gamma x).$$

   The boundary condition

$$X(0) = 0 \implies A = 0.$$

   The boundary condition

$$X'(l) + X(l) = 0 \implies B\gamma\cosh(\gamma l) + B\sinh(\gamma l) = 0.$$

   Now, $\gamma\cosh(\gamma l)+\sinh(\gamma l) \neq 0$, because the curves $f(\gamma) = \gamma$ and $g(\gamma) = -\tanh(\gamma l)$ do not intersect for $\gamma \neq 0$. Therefore, in order to satisfy the condition above, we need $B = 0$. But, this implies $X(x) \equiv 0$. Again, the zero function is not an eigenfunction. Therefore, we have no negative eigenvalues.

2. (18 points) Consider the following initial/boundary value problem,

$$(*)\begin{cases} u_{tt} - u_{xx} + u = f(x,t) & 0 < x < l \\ u(x,0) = 0 \\ u_t(x,0) = 0 \\ u(0,t) = 0 \\ u_x(l,t) + u(l,t) = 0 \end{cases}$$

(a) (10 points) Let $\{X_n(x), \lambda_n\}$ denote the eigenfunctions and eigenvalues found in the previous problem. Solve (*) in terms of X_n and λ_n. (You do not need to use your answer from problem 1.)

Answer: First, we will solve the homogeneous problem above, and then we will use Duhamel's principle. We can rewrite this equation as a system as follows. Letting $v = u_t$, we get the following system,

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ \partial_x^2 - 1 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

If $S(t)$ is the solution operator associated with the homogeneous equation, then the solution of the inhomogeneous problem (*) will be given by

$$\int_0^t S(t-s)F(s)\,ds$$

using the fact that the initial conditions are zero. Therefore, we will look for $S(t)$. To do so, we will use separation of variables.

Consider the homogeneous problem. Looking for a solution of the form $u(x,t) = X(x)T(t)$, we are led to the equation

$$XT'' - X''T + XT = 0.$$

Dividing by XT, we get

$$\frac{T''}{T} + 1 = \frac{X''}{X} = -\lambda.$$

By assumption, the eigenvalues and eigenfunctions of

$$\begin{cases} X'' = -\lambda X & 0 < x < l \\ X(0) = 0 & X'(l) + X(l) = 0. \end{cases}$$

are given by $X_n(x)$ and $\lambda_n(x)$. Now we look for solutions of

$$\frac{T_n''}{T_n} + 1 = -\lambda$$

or rewritten as

$$T_n'' + (1 + \lambda_n)T_n = 0.$$

By problem 1, we know all eigenvalues are positive. Therefore, the solutions of this equation are given by

$$T_n(t) = A\cos(\sqrt{1+\lambda_n}t) + B\sin(\sqrt{1+\lambda_n}t).$$

Let

$$u(x,t) = \sum_{n=1}^{\infty}[A_n\cos(\sqrt{1+\lambda_n}t) + B_n\sin(\sqrt{1+\lambda_n}t)]X_n(x).$$

For general initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, our coefficients will be given by

$$\begin{aligned} A_n &= \frac{\langle \phi, X_n\rangle|_{[0,l]}}{\langle X_n, X_n\rangle|_{[0,1]}} \\ \sqrt{1+\lambda_n}B_n &= \frac{\langle \psi, X_n\rangle|_{[0,l]}}{\langle X_n, X_n\rangle|_{[0,l]}}. \end{aligned}$$

We conclude that the solution of (*) is given by

$$\boxed{u(x,t) = \int_0^t \sum_{n=1}^{\infty} D_n(s)\sin(\sqrt{1+\lambda_n}t)X_n(x)}$$

where

$$\boxed{D_n(s) = \frac{\langle f(x,s), X_n(x)\rangle|_{[0,l]}}{\sqrt{1+\lambda_n}\langle X_n, X_n\rangle|_{[0,l]}}.}$$

(b) (8 points) Prove uniqueness of the solution to (*).

Answer: Suppose there were two solutions u and v. Let $w = u - v$. Then w satisfies

$$\begin{cases} w_{tt} - w_{xx} + w = 0 \quad 0 < x < l \\ w(x,0) = 0 \\ w_t(x,0) = 0 \\ w(0,t) = 0 \\ w_x(l,t) + w(l,t) = 0. \end{cases}$$

Multiply this equation by w_t and integrate over $[0,l]$ with respect to x. We have

$$\begin{aligned} 0 &= \int_0^l w_t[w_{tt} - w_{xx} + w]\,dx \\ &= \int_0^l \frac{1}{2}\partial_t(w_t^2) + w_{xt}w_x + \frac{1}{2}\partial_t(w^2)\,dx - w_t w_x|_{x=0}^{x=l} \\ &= \frac{1}{2}\partial_t\left(\int_0^l [w_t^2 + w_x^2 + w^2]\,dx\right) + w_t(l,t)w(l,t) + w_t(0,t)w_x(0,t) \\ &= \frac{1}{2}\partial_t\left(\int_0^l [w_t^2 + w_x^2 + w^2]\,dx + w^2(l,t)\right) \end{aligned}$$

using the fact that $w(0,t) = 0$ implies $w_t(0,t) = 0$ and $w_x(l,t) = -w(l,t)$. Integrating this equation with respect to t, we have

$$\int_0^l [w_t^2(x,t) + w_x^2(x,t) + w^2(x,t)]\,dx + w^2(l,t)$$
$$= \int_0^l [w_t^2(x,0) + w_x^2(x,0) + w^2(x,0)]\,dx + w^2(l,0).$$

By the initial conditions, we conclude that the right-hand side is zero, and, therefore, the left-hand side is identically zero. Therefore, we conclude that $w^2(x,t) \equiv 0$ for all $x \in [0,l]$, which implies that $w(x,t) \equiv 0$, and, thus, $u = v$.

3. (8 points) Use the method of reflection to show that the solution of

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = 0 = u(l,t) \end{cases}$$

is periodic in t. That is, show that there exists $p \in \mathbb{R}$ such that for all $x \in (0,l)$, $u(x,t) = u(x,t+p)$. Specifically, find the period p.

Answer: Since ϕ_{ext} is $2L$-periodic,

$$\phi_{\text{ext}}(x+ct) = \phi_{\text{ext}}(x+ct+2L) = \phi_{\text{ext}}(x + c(t+2L/c))$$

so ϕ_{ext} is periodic in t with period $2L/c$. Since ψ_{ext} is odd and $2L$-periodic, the integral of ψ_{ext} over any interval of length $2L$ is zero. Thus

$$\begin{aligned} u(x,t+2L/c) &= \frac{1}{2}[\phi_{\text{ext}}(x+ct+2L) + \phi_{\text{ext}}(x-ct-2L)] + \frac{1}{2c}\int_{x-ct-2L}^{x+ct+2L} \psi_{\text{ext}}(y)\,dy \\ &= \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct-2L}^{x-ct} \psi_{\text{ext}}(y)\,dy \\ &+ \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(y)\,dy + \frac{1}{2c}\int_{x+ct}^{x+ct+2L} \psi_{\text{ext}}(y)\,dy \\ &= \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(y)\,dy \\ &= u(x,t) \end{aligned}$$

so u is periodic in t with period $2L/c$.

4. (14 points) Consider the following equation for $u = u(x,y,z)$,

$$4u_{xy} + 4u_{xz} + 4u_{yz} = 0.$$

(a) (6 points) Show the equation is hyperbolic.

Answer: The equation can be rewritten as

$$\sum_{i,j=1}^{3} a_{ij} u_{x_i x_j} = 0$$

where $A = (a_{ij})$ is the symmetric matrix,

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}.$$

The eigenvalues of A are given by the roots of the characteristic equation $\det(A - \lambda I) = 0$. This equation is given by $(\lambda+2)^2(\lambda-4) = 0$. Therefore, we see that $\lambda = -2$ and $\lambda = 4$ are the eigenvalues of this equation. As none of them are zero, and one has the opposite sign of the other two, the equation is hyperbolic.

(b) (8 points) Make a change of variables to reduce it to

$$\alpha_1 u_{\xi_1\xi_1} + \alpha_2 u_{\xi_2\xi_2} + \alpha_3 u_{\xi_3\xi_3} = 0$$

where $\alpha_i \in \mathbb{R}$. In particular, find a matrix B such that defining

$$\xi \equiv \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = B \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

our equation can written in the form above. Determine the values for α_i.

Answer: We will define the matrix B as $B = S^T$ where S is the matrix whose column vectors are orthonormal eigenvectors of A. First, for $\lambda_1 = -2$, we see that

$$A - \lambda_1 I = A + 2I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(where $\to$ denotes row-reduction). An orthonormal basis for the eigenspace of $A - \lambda_1 I$ is given by

$$\left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

For $\lambda_2 = 4$,we see that

$$A - \lambda_2 I = A - 4I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

A basis for the eigenspace of $A - \lambda_2 I$ is given by

$$\left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

Let

$$S = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{-2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Now let $B = S^T$. The coefficients $\alpha_i = -2, -2, 4$.

5. (10 points) Consider the following initial-value problem,

$$\begin{cases} u_t + [f(u)]_x = 0 \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

Assume f is uniformly convex and ϕ is a smooth function. Show that if $\phi(x) = -x$, then $|u_x| \to +\infty$ in finite time.

Answer: The characteristic equations are given by

$$\frac{dt}{ds} = 1 \qquad t(r,0) = 0$$
$$\frac{dx}{ds} = f'(z) \qquad x(r,0) = r$$
$$\frac{dz}{ds} = 0 \qquad z(r,0) = \phi(r).$$

Solving this system, we get

$$t = s$$
$$x = f'(\phi(r))s + r$$
$$z = \phi(r).$$

We arrive at the following implicit equation for $u(x,t)$,

$$u(x,t) = \phi(x - f'(u)t).$$

Differentiating this equation with respect to x, we see that u_x is given by

$$u_x = \frac{\phi'(x - f'(u)t)}{1 + \phi'(x - f'(u)t)f''(u)t}.$$

Now if $\phi(x) = -x$, then $\phi'(x) = -1$, which implies

$$u_x = \frac{-1}{1 - f''(u)t}.$$

Now if $1 - f''(u)t = 0$ in some finite time, then $|u_x| \to +\infty$. By the uniform convexity assumption, we know that there exists θ such that $f''(u) \geq \theta > 0$ for all u. Therefore, $0 < 1/f''(u) \leq C$. Therefore, there exists a time $t < +\infty$ such that $t = 1/f''(u)$, and u_x blows up.

6. (14 points) Consider the system

$$U_t + AU_x = 0$$

where

$$A = \begin{bmatrix} 1 & 2e^x \\ 2e^{-x} & 1 \end{bmatrix}.$$

(a) (6 points) Show that this system is hyperbolic.

Answer: We need to show that the matrix A is diagonalizable. First, we compute the eigenvalues of A. The eigenvalues are given by the roots of $\det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = 0$. The roots of this equation are given by $\lambda = 3, -1$. Since the roots are distinct, we know the eigenvectors associated with them are linearly independent, and, thus A is diagonalizable. Therefore, this system is hyperbolic.

(b) (8 points) Solve the initial-value problem,

$$\begin{cases} U_t + \begin{bmatrix} 1 & 2e^x \\ 2e^{-x} & 1 \end{bmatrix} U_x = 0 \\ U(x,0) = \begin{bmatrix} \sin(x) \\ 0 \end{bmatrix}. \end{cases}$$

Answer: We begin by diagonalizing the system. For $\lambda_1 = 3$,

$$A - \lambda_1 I = \begin{bmatrix} -2 & 2e^x \\ 2e^{-x} & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -e^x \\ 0 & 0 \end{bmatrix}.$$

Therefore, an eigenvector for $\lambda_1 = 3$ is given by

$$\left\{ \begin{bmatrix} e^x \\ 1 \end{bmatrix} \right\}.$$

For $\lambda_2 = -1$,

$$A - \lambda_2 I = \begin{bmatrix} 2 & 2e^x \\ 2e^{-x} & 2 \end{bmatrix} \to \begin{bmatrix} 1 & e^x \\ 0 & 0 \end{bmatrix}.$$

Therefore, an eigenvector for $\lambda_2 = -1$ is given by

$$\left\{ \begin{bmatrix} e^x \\ -1 \end{bmatrix} \right\}.$$

Let

$$Q = \begin{bmatrix} e^x & e^x \\ 1 & -1 \end{bmatrix}.$$

Then

$$Q^{-1} = \frac{1}{2e^x} \begin{bmatrix} 1 & e^x \\ 1 & -e^x \end{bmatrix}.$$

And, we have

$$Q^{-1}AQ = \Lambda = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Plugging in for $A = Q\Lambda Q^{-1}$, we have

$$U_t + Q\Lambda Q^{-1} U_x = 0.$$

Multiplying by Q^{-1}, we have

$$Q^{-1}U_t + \Lambda Q^{-1} U_x = 0.$$

This can be rewritten as

$$(Q^{-1}U)_t + \Lambda(Q^{-1}U)_x - \Lambda(Q^{-1})_x U = 0.$$

Now let $V = Q^{-1}U$. Then, we arrive at

$$V_t + \Lambda V_x = \Lambda(Q^{-1})_x U.$$

Now

$$\Lambda(Q^{-1})_x U = \frac{1}{2}\Lambda \begin{bmatrix} -e^{-x} & 0 \\ -e^{-x} & 0 \end{bmatrix} U = \frac{1}{2}\begin{bmatrix} -3e^{-x}u_1 \\ -e^{-x}u_1 \end{bmatrix}.$$

Therefore, our equation for V becomes

$$V_t + \Lambda V_x = \frac{1}{2}\begin{bmatrix} -3e^{-x}u_1 \\ -e^{-x}u_1 \end{bmatrix}.$$

This system decouples into two initial-value problems for inhomogeneous transport equations,

$$\begin{cases} (v_1)_t + 3(v_1)_x = -\frac{3}{2}e^{-x}u_1 \\ v_1(x,0) = \frac{1}{2}e^{-x}\sin(x) \end{cases}$$

and

$$\begin{cases} (v_2)_t - (v_1)_x = -e^{-x}u_1 \\ v_2(x,0) = \frac{1}{2}e^{-x}\sin(x). \end{cases}$$

The solutions of these equations are given by

$$v_1(x,t) = \frac{1}{2}e^{-(x-3t)}\sin(x-3t) - \frac{3}{2}\int_0^t e^{-(x-(t-s))}u_1(x-(t-s),s)\,ds$$
$$v_2(x,t) = \frac{1}{2}e^{-(x+t)}\sin(x+t) - \int_0^t e^{-(x-(t-s))}u_1(x-(t-s),s)\,ds.$$

Finally, the solution of our original system is given (implicitly) by $U = QV$.

7. (10 points) Answer the following short-answer questions.

(a) (5 points) State the definition of a kth-order quasilinear partial differential equation for a function $u : \mathbb{R}^n \to \mathbb{R}$.

Answer: We say a kth-order partial differential equation is quasilinear if it is nonlinear, but not semilinear, and can be written in the following form,

$$\sum_{|\alpha|=k} a_\alpha(x, u, \dots, D^{k-1}u)D^\alpha u + a_0(x, u, \dots, D^k u) = 0,$$

where $D^j u$ denotes the collection of all partial derivatives of order j and α is a multi-index.

(b) (5 points) Give an example of an initial-value problem for a first-order semilinear equation in which the initial data is smooth, but the solution blows up in finite time.

Answer: Consider the following initial-value problem,

$$\begin{cases} u_t + u_x = u^2 \\ u(x,0) = \sin(x). \end{cases}$$

Now the characteristics are given by

$$\begin{aligned} \frac{dt}{ds} &= 1 \qquad t(r,0) = 0 \\ \frac{dx}{ds} &= 1 \qquad x(r,0) = r \\ \frac{dz}{ds} &= z^2 \qquad z(r,0) = \sin(r). \end{aligned}$$

The solutions of this system are given by

$$\begin{aligned} t &= s \\ x &= s + r \\ \frac{-1}{z} &= s - \frac{1}{\sin(r)}. \end{aligned}$$

Solving this system, we find that our solution is given by

$$u(x,t) = \frac{\sin(x-t)}{1 - t\sin(x-t)}.$$

We see that at $t = 1$, $x = 1 + (2n+1)\pi/2$, the solution will blow up.

8. (18 points) Determine whether each of the following statements is True or False. Provide reasons for your answers.

(a) (3 points) Assume f is uniformly convex. Consider

$$(*) \begin{cases} u_t + [f(u)]_x = 0 \quad x \in \mathbb{R}, t > 0 \\ u(x,0) = \phi(x). \end{cases}$$

Suppose

$$\phi(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0 \end{cases}$$

Suppose u is a weak solution of (*). Then $u(x,t)$ is continuous for $t > 0$.

Answer: $\boxed{False}$ Suppose $u^- > u^+$. Let

$$u(x,t) = \begin{cases} u^- & x < \sigma t \\ u^+ & x > \sigma t \end{cases}$$

where $\sigma = [f(u)]/[u]$. Then u is a weak solution, but u is not continuous.

(b) (3 points) Let Ω be a bounded subset of $\mathbb{R}^n$. All linearly independent eigenfunctions of

$$\begin{cases} -\Delta X = \lambda X & x \in \Omega \subset \mathbb{R}^n \\ X(x) = 0 \text{ for } x \in \partial\Omega \end{cases}$$

are orthogonal.

Answer: $\boxed{False}$ Consider the case when Ω is a rectangle in $\mathbb{R}^2$. In particular, consider $\Omega = (0,\pi) \times (0,\pi)$. Then the eigenvalues are given by $\lambda_{nm} = n^2 + m^2$ with eigenfunctions $X_{nm}(x,y) = \sin(nx)\sin(my)$. In particular, we see that $\lambda = 5$ is an eigenvalue with multiplicity two. We note that $X_{12} = \sin(x)\sin(2y)$ and $X_{21} = \sin(2x)\sin(y)$ are both eigenfunctions associated with $\lambda = 5$. In addition, any linear combination of these eigenfunctions is an eigenfunction with eigenvalue $\lambda = 5$. But, clearly, not all of these eigenfunctions are orthogonal.

(c) (3 points) Suppose u is a solution of

$$\begin{cases} u_{tt} - u_{xx} = 1 & x \in \mathbb{R}^3, t > 0 \\ u(x,0) = 0 & x \in \mathbb{R}^3 \\ u_t(x,0) = 0 & x \in \mathbb{R}^3. \end{cases}$$

Then $u(x,t) \neq 0$ for all $t > 0$, all $x \in \mathbb{R}^3$.

Answer: $\boxed{True}$ The solution is given by

$$\begin{aligned} u(x,t) &= \int_0^t (t-s) \fint_{\partial B(x,t-s)} dS(y)\, ds \\ &= \int_0^t (t-s)\, ds \\ &= ts - \frac{s^2}{2}\bigg|_{s=0}^{s=t} = \frac{t^2}{2} \neq 0 \end{aligned}$$

for all $t > 0$, all $x \in \mathbb{R}^3$.

(d) (3 points) Let $f(x) = 1$ for $x \in [0,l]$. The Fourier sine series for f will converge uniformly to f on $[0,1]$.

Answer: $\boxed{False}$ At $x = 0$, $\sin(n\pi x/l) = 0$ implies the Fourier sine series for f is zero at $x = 0$, but $f(0) = 1$. Therefore, the series cannot converge uniformly.

(e) (3 points) If A_i is an $m \times m$, constant-coefficent, diagonalizable matrix for $i = 1, \dots, n$, then

$$U_t + \sum_{i=1}^{n} A_i U_{x_i} = 0$$

is a hyperbolic system.

Answer: $\boxed{False}$ The matrices A_i being diagonalizable does not imply that $A(\xi) \equiv \sum_{i=1}^{n} A_i \xi_i$ is diagonalizable for all $\xi \in \mathbb{R}^n$. In particular, consider $m = n = 2$ and let

$$A_1 = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then A_1 and A_2 are both diagonalizable because they both have two distinct eigenvalues. But, letting $\xi = (1, -1)$, we see that

$$A(\xi) = A_1 - A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

which is not diagonalizable.

(f) (3 points) Consider the initial-value problem for the hyperbolic equation

$$\begin{cases} u_{tt} - \sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} = 0 & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

where the eigenvalues of $A = (a_{ij})$ are all positive. If ϕ and ψ have compact support, then u has compact support.

Answer: $\boxed{True}$ By a change of variables this equation can be written as the wave equation in $\mathbb{R}^n$,

$$u_{tt} - \Delta u = 0.$$

We know that if the initial data has compact support, then the solution has compact support (in x).