3 Conservation Laws

3.1 Motivation

Example 1. *(Burgers' Equation)* Consider the initial-value problem for Burgers' equation, a first-order quasilinear equation of the form

$$\begin{cases} u_t + uu_x = 0 \\ u(x,0) = \phi(x). \end{cases}$$

This equation models wave motion, where $u(x,t)$ is the height of the wave at point x, time t. As described earlier, if $\phi'(x) < 0$, we may have projected characteristic curves intersecting, resulting in a difficulty in defining the solution beyond the point of intersection. This phenomenon occurs as a result of wave breaking. In order to define solutions which "solve" this problem, we need to determine how to define solutions beyond this point where the projected characteristic curves intersect. In particular, we need to allow for solutions which may not even be continuous! In what sense can a function which is not even continuous, let alone differentiable, satisfy a *differential* equation? We will discuss that momentarily. First, we present another example.

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Example 2. *(Traffic Flow Problem)* Consider a street starting at point x_1 and ending at point x_2. Let $u(x,t)$ be the density of cars at point x, time t. Therefore, the total number of cars between point x_1 and x_2 at time t can be represented by

$$\int_{x_1}^{x_2} u(x,t)\,dx.$$

Now the rate of change in the number of cars between points x_1 and x_2 at time t is given by

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t))$$

where f represents the flow rate onto and off the street. Assuming u and f are continuously differentiable functions, we see that

$$\int_{x_1}^{x_2} u_t(x,t)\,dx = f(u(x_1,t)) - f(u(x_2,t)),$$

and, therefore,

$$\frac{1}{x_2 - x_1}\int_{x_1}^{x_2} u_t(x,t)\,dx = \frac{f(u(x_1,t)) - f(u(x_2,t))}{x_2 - x_1}.$$

Taking the limit as $x_2 \to x_1$, we get

$$u_t = -[f(u)]_x.$$

Therefore, we can say that the density of cars at point x at time t satisfies the PDE:

$$u_t + [f(u)]_x = 0$$

for some smooth function f. However, this is assuming the density of cars is a continuous function. We would like to derive some sort of notion to say that a function u which is not even differentiable will "solve" the PDE.

$\diamond$

3.2 Weak Solutions

In this section we introduce the notion of a solution u to a partial differential equation where u may not even be a differentiable function. This type of solution will be known as a *weak* solution of a PDE. First, however, we define the notion of a *strong* solution of a partial differential equation. Consider the initial-value problem for the k-th order PDE,

$$\begin{cases} F(\vec{x}, u, Du, \ldots, D^k u) = 0 \quad x \in \mathbb{R}^n, t > 0 \\ u(\vec{x}, 0) = \phi(\vec{x}) \end{cases} \tag{3.1}$$

We say u is a **strong** or **classical** solution of (3.1) if u is k times continuously differentiable and u satisfies (3.1). This is probably the idea you have had in mind when you think of solving a PDE. However, as described in the examples above, sometimes a classical solution does not always exist, or you may want to allow for "solutions" which are not differentiable, or even continuous. What do we mean by this?

In this section we will define the notion of a *weak* solution of a first-order, quasilinear initial-value problem of the form

$$\begin{cases} u_t + [f(u)]_x = 0 \qquad x \in \mathbb{R}, t \geq 0 \\ u(x, 0) = \phi(x). \end{cases} \tag{3.2}$$

Before doing so, however, we give some preliminary definitions. We say a subset of $\mathbb{R}^n$ is **compact** if it is closed and bounded. We say a function $v : \mathbb{R}^n \to \mathbb{R}$ has **compact support** if $v \equiv 0$ outside some compact set. Now, we say that u is a **weak solution** of (3.2) if

$$\int_0^\infty \int_{-\infty}^\infty [u v_t + f(u) v_x] \, dx \, dt + \int_{-\infty}^\infty \phi(x) v(x, 0) \, dx = 0 \tag{3.3}$$

for all smooth functions $v \in C^\infty(\mathbb{R} \times [0, \infty))$ with compact support.

Notice that a function u need not be differentiable or even continuous for it to satisfy (3.3). Functions u which satisfy (3.3) may not be classical solutions of (3.2). However, they should satisfy (3.2) in some sense. Where did (3.3) come from? So far, we have just made a definition. We will now prove that if u is a strong solution of (3.2), then u is a weak solution of (3.2); that is, u will satisfy (3.3). In this sense, condition (3.3) is a natural extension of the notion of a "solution" to (3.2).

Theorem 3. *If u is a strong solution of* (3.2), *then u is a weak solution of* (3.2).

Proof. If u is a classical solution of (3.2), then u is continuously differentiable and

$$\begin{cases} u_t + [f(u)]_x = 0, \qquad t \geq 0 \\ u(x, 0) = \phi(x). \end{cases}$$

In addition, for any smooth function $v : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ with compact support,

$$\int_0^\infty \int_{-\infty}^\infty [u_t + [f(u)]_x] v \, dx \, dt = 0. \tag{3.4}$$

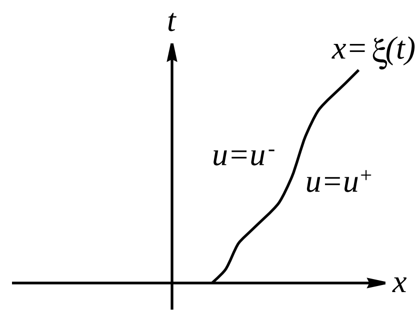
Integrating (3.4) by parts and using the fact that v vanishes at infinity, we see that

$$\int_0^\infty \int_\infty^\infty [uv_t + f(u)v_x] \, dx \, dt + \int_{-\infty}^\infty \phi(x) v(x, 0) \, dx = 0.$$

But this is true for all functions $v \in C^\infty(\mathbb{R} \times [0, \infty))$ with compact support. Therefore, u is a weak solution of (3.2).

$\square$

As mentioned above, the notion of weak solution allows for solutions u which need not even be continuous. However, weak solutions u have some restrictions on types of discontinuities, etc. For example, suppose u is a weak solution of (3.2) such that u is discontinuous across some curve $x = \xi(t)$, but u is smooth on either side of the curve. Let $u^-(x, t)$ be the limit of u approaching (x, t) from the left and let $u^+(x, t)$ be the limit of u approaching (x, t) from the right. We claim that the curve $x = \xi(t)$ cannot be arbitrary, but rather there is a relation between $x = \xi(t)$, u^- and u^+.



Theorem 4. *If u is a weak solution of (3.2) such that u is discontinuous across the curve $x = \xi(t)$ but u is smooth on either side of $x = \xi(t)$, then u must satisfy the condition*

$$\frac{f(u^-) - f(u^+)}{u^- - u^+} = \xi'(t) \tag{3.5}$$

across the curve of discontinuity, where $u^-(x, t)$ is the limit of u approaching (x, t) from the left and $u^+(x, t)$ is the limit of u approaching (x, t) from the right.

Proof. If u is a weak solution of (3.2), then

$$\int_0^\infty \int_{-\infty}^\infty [uv_t + f(u)v_x] \, dx \, dt + \int_{-\infty}^\infty \phi(x) v(x, 0) \, dx = 0$$

for all smooth functions $v \in C^\infty(\mathbb{R} \times [0, \infty))$ with compact support. Let v be a smooth function such that $v(x, 0) = 0$, and break up the first integral into the regions Ω^-, Ω^+ where

$$\Omega^- \equiv \{(x, t) : 0 < t < \infty, \, -\infty < x < \xi(t)\}$$
$$\Omega^+ \equiv \{(x, t) : 0 < t < \infty, \, \xi(t) < x < +\infty\}.$$

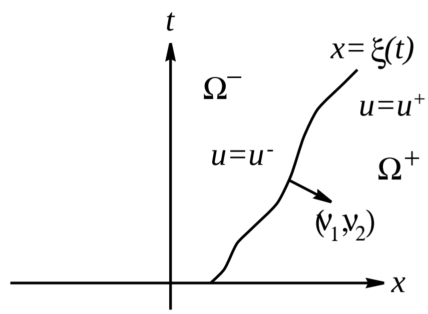
Therefore,

$$\begin{aligned} 0 &= \int_0^\infty \int_{-\infty}^\infty [uv_t + f(u)v_x]\,dx\,dt + \int_{-\infty}^\infty \phi(x)v(x,0)\,dx \\ &= \iint_{\Omega^-} [uv_t + f(u)v_x]\,dx\,dt + \iint_{\Omega^+} [uv_t + f(u)v_x]\,dx\,dt. \end{aligned} \tag{3.6}$$

Combining the Divergence Theorem with the fact that v has compact support and $v(x,0) = 0$, we have

$$\iint_{\Omega^-} [uv_t + f(u)v_x]\,dx\,dt = -\iint_{\Omega^-} [u_t + (f(u))_x]v\,dx\,dt + \int_{x=\xi(t)} [u^- v\nu_2 + f(u^-)v\nu_1]\,ds \tag{3.7}$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal to Ω^-.



Similarly, we see that

$$\iint_{\Omega^+} [uv_t + f(u)v_x]\,dx\,dt = -\iint_{\Omega^+} [u_t + (f(u))_x]v\,dx\,dt - \int_{x=\xi(t)} [u^+ v\nu_2 + f(u^+)v\nu_1]\,ds \tag{3.8}$$

By assumption, u is a weak solution of

$$u_t + [f(u)]_x = 0$$

and u is smooth on either side of $x = \xi(t)$. Therefore, u is a strong solution on either side of the curve of discontinuity. Consequently, we see that

$$\iint_{\Omega^-} [u_t + (f(u))_x]v\,dx\,dt = 0 = \iint_{\Omega^+} [u_t + (f(u))_x]v\,dx\,dt.$$

Combining this fact with (3.6), (3.7) and (3.8), we see that

$$\int_{x=\xi(t)} [u^- v\nu_2 + f(u^-)v\nu_1]\,ds - \int_{x=\xi(t)} [u^+ v\nu_2 + f(u^+)v\nu_1]\,ds = 0.$$

Since this is true for all smooth functions v, we have

$$u^-\nu_2 + f(u^-)\nu_1 = u^+\nu_2 + f(u^+)\nu_1,$$

which implies

$$\frac{f(u^-) - f(u^+)}{u^- - u^+} = -\frac{\nu_2}{\nu_1}.$$

Now the curve $x = \xi(t)$ has slope given by the negative reciprocal of the normal to the curve; that is,

$$\frac{dt}{dx} = \frac{1}{\xi'(t)} = -\frac{\nu_1}{\nu_2}.$$

Therefore,

$$\xi'(t) = -\frac{\nu_2}{\nu_1} = \frac{f(u^-) - f(u^+)}{u^- - u^+}.$$

Therefore, if the solution u has a discontinuity along a curve $x = \xi(t)$, the solution u and the curve $x = \xi(t)$ must satisfy the condition

$$\frac{f(u^-) - f(u^+)}{u^- - u^+} = \xi'(t).$$

$\square$

For shorthand notation, we define

$$\begin{aligned}
&[u] = u^- - u^+ \\
&[f(u)] = f(u^-) - f(u^+) \\
&\sigma = \xi'(t)
\end{aligned}$$

We call $[u]$ and $[f(u)]$ the **jumps** of u and $f(u)$ across the discontinuity curve and σ the **speed** of the curve of discontinuity. Therefore, if u is a weak solution with discontinuity along a curve $x = \xi(t)$, the solution must satisfy

$$[f(u)] = \sigma[u] \tag{3.9}$$

where $\sigma = \xi'(t)$. This is called the **Rankine-Hugoniot jump condition**.

Example 5. Consider Burgers' equation

$$\begin{cases} u_t + uu_x = 0, & t \geq 0 \\ u(x,0) = \phi(x), \end{cases} \tag{3.10}$$

where the initial condition $\phi(x)$ satisfies

$$\phi(x) = \begin{cases} 1 & \text{for } x < 0 \\ 1 - x & \text{for } 0 < x < 1 \\ 0 & \text{for } x > 1. \end{cases} \tag{3.11}$$

Notice that we can write this equation in standard form as

$$u_t + \left[\frac{u^2}{2}\right]_x = 0.$$

In trying to solve this equation using the method of characteristics, our characteristic equations are given by

$$\begin{aligned}\frac{dt}{ds} &= 1\\ \frac{dx}{ds} &= z\\ \frac{dz}{ds} &= 0\end{aligned}$$

with initial conditions

$$\begin{aligned}t(r,0) &= 0\\ x(r,0) &= r\\ z(r,0) &= \phi(r).\end{aligned}$$

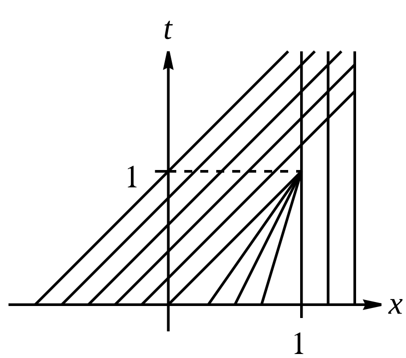
We see that the solution is given by

$$\begin{aligned}t &= s\\ x &= \phi(r)s + r\\ z &= \phi(r).\end{aligned}$$

From these solutions, we arrive at an implicit solution for (3.10) as

$$u = \phi(x - ut).$$

Using the fact that $\frac{dz}{ds} = 0$, we see that u is constant along the projected characteristic curves, $x = \phi(r)t + r$. We parametrize Γ by r such that $\Gamma = \{(r, 0)\}$. We see that for $r < 0$, $\phi(r) = 1$ and therefore, these projected characteristic curves are given by $x = t + r$ for $-\infty < r < 0$, and $u(x,t) = 1$ along these curves. For $0 < r < 1$, $\phi(r) = 1 - r$ and therefore, these projected characteristic curves are given by $x = (1-r)t + r$ for $0 < r < 1$. Moreover, along these curves $u(x,t) = z(r,s) = 1 - r = (1-x)/(1-t)$. Finally, for $r > 1$, $\phi(r) = 0$. Therefore, the projected characteristic curves are given by $x = r$ for $r > 1$, and $u = 0$ along these curves.



For $t \leq 1$, our solution is defined as

$$u(x,t) = \begin{cases} 1 & x < t \\ \dfrac{1-x}{1-t} & t < x < 1 \\ 0 & x > 1. \end{cases} \tag{3.12}$$

However, notice that the curves intersect at $t = 1$. Beyond that time t, the different projected characteristics are asking for our solution u to satisfy different conditions. This cannot happen. We no longer have a classical solution. Instead, let's look for a weak solution of (3.10) for $t \geq 1$ which satisfies (3.11).

From Theorem 4, a weak solution must satisfy the Rankine-Hugoniot jump condition discussed above. That is, we need

$$[f(u)] = \sigma[u],$$

or more specifically,

$$\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = \xi'(t)[u^- - u^+].$$

Notice that the initial data for $x < 0$ wants $u = 1$, while the initial data for $x > 1$ wants $u = 0$ for $t \geq 1$. Let's try to make a compromise by defining a curve $x = \xi(t)$ such that $u = 1$ to the left of the curve and $u = 0$ to the right of the curve. In other words, let $u^- = 1$ and $u^+ = 0$. Now our curve $x = \xi(t)$ is defined for us by the Rankine-Hugoniot jump condition. In particular, we have

$$\xi'(t) = \frac{1}{2}.$$

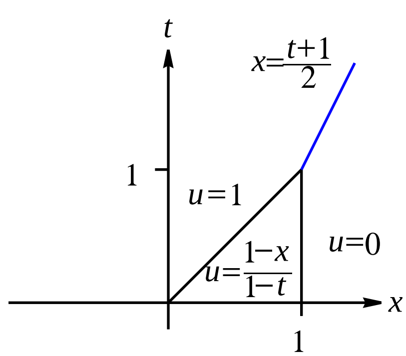
In addition, we want our curve $x = \xi(t)$ to contain the point $(x,t) = (1,1)$. Therefore, our curve must be given by $(x-1) = \frac{1}{2}(t-1)$ or $x = \frac{t+1}{2}$.

Therefore, for $t \geq 1$, let

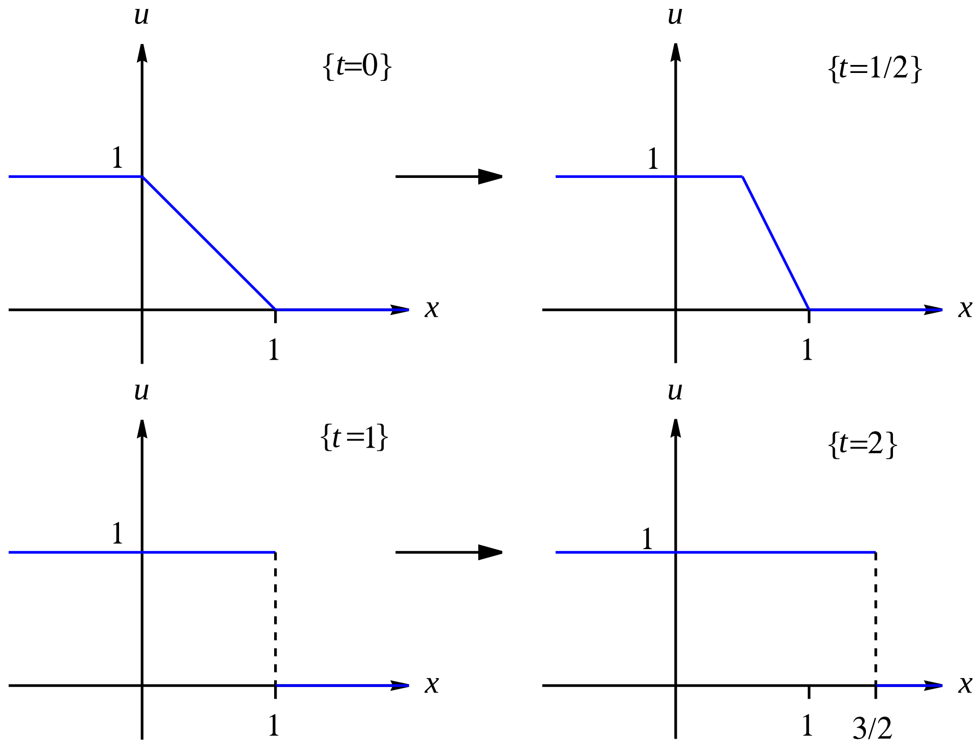
$$u(x,t) = \begin{cases} 1 & \text{if } x < \dfrac{t+1}{2} \\ 0 & \text{if } x > \dfrac{t+1}{2}. \end{cases} \tag{3.13}$$

Now u defined by (3.13) is a classical solution of (3.10) on either side of the curve $x(t) = \frac{t+1}{2}$ and u satisfies the Rankine-Hugoniot jump condition along the curve of discontinuity. Therefore, $u(x,t)$ defined by (3.13) is a weak solution of (3.10) for $t \geq 1$.

In summary, our solution of (3.10) with initial condition (3.11) is given by (3.12) for $t \leq 1$ and (3.13) for $t \geq 1$.



Below we show a "movie" of our solution.



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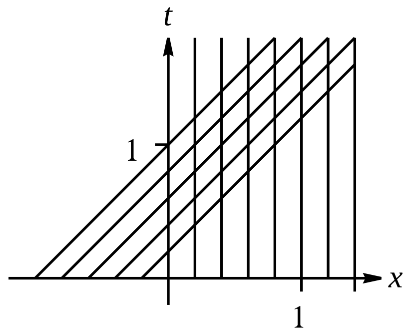
Example 6. Consider Burger's equation again,

$$\begin{cases} u_t + uu_x = 0 & t \geq 0 \\ u(x,0) = \phi(x). \end{cases} \tag{3.14}$$

But this time impose the initial condition,

$$\phi(x) = \begin{cases} 1 & \text{for } x < 0 \\ 0 & \text{for } x > 0. \end{cases} \tag{3.15}$$

As before, u is constant along the projected characteristic curves given by $x = \phi(r)t + r$. If $r < 0$, then $\phi(r) = 1$ which implies the projected characteristic curves are $x = t + r$ for $r < 0$ and the solution u should equal 1 along those curves. But, also, for $r > 0$, $\phi(r) = 0$ which means the projected characteristic curves are given by $x = r$ for $r > 0$ and the solution u should equal 0 along these curves.



Clearly, this is a contradiction and we can't hope to find any continuous solution which solves this problem. Again, we look for a weak solution, by looking for a piecewise continuously differentiable function which satisfies the Rankine Hugoniot jump condition. We want to find a curve $x = \xi(t)$ such that $u^- = 1$ to the left of the curve and $u^+ = 0$ to the right of the curve. We need

$$[f(u)] = \sigma[u].$$

That is, we need

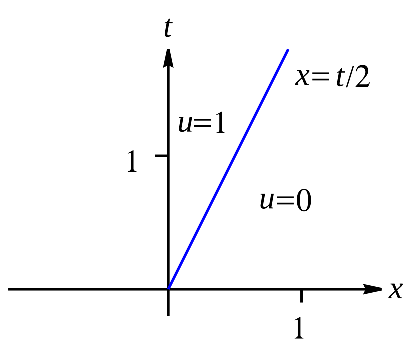
$$\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = \xi'(t)[u^- - u^+].$$

Substituting in for u^- and u^+, we get

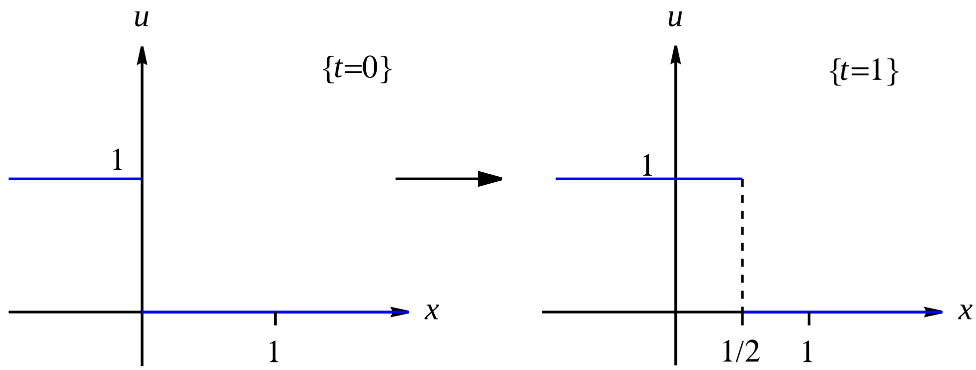
$$\xi'(t) = \frac{1}{2}.$$

In addition, we want the curve $x = \xi(t)$ to contain the point $(x, t) = (0, 0)$. Therefore, our curve of discontinuity must be given by $x = \frac{1}{2}t$. Therefore, our weak solution of (3.14) satisfying (3.15) is given by

$$u(x,t) = \begin{cases} 1 & \text{for } x < \dfrac{t}{2} \\ 0 & \text{for } x > \dfrac{t}{2}. \end{cases}$$



Below we show a "movie" of our solution.



$\diamond$

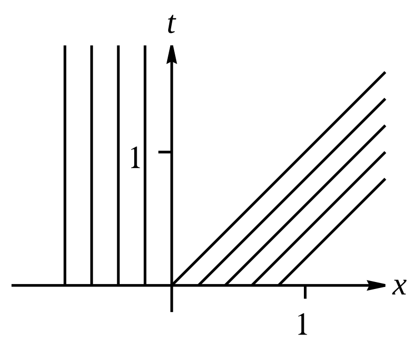
Example 7. We consider Burgers' equation once again,

$$\begin{cases} u_t + uu_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases} \tag{3.16}$$

but now impose the initial condition

$$\phi(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x > 0. \end{cases} \tag{3.17}$$

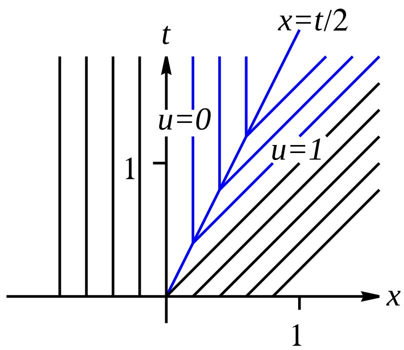
Looking at our characteristics, we see that u should be constant along the projected characteristic curves, $x = \phi(r)t + r$. Here, if $r < 0$, then $\phi(r) = 0$ and therefore, $x = r$. If $r > 0$, then $\phi(r) = 1$ and therefore, $x = t + r$. Consequently, we have no crossing of characteristics. However, we still have a problem. In fact, we have a region on which we don't have enough information! How should we define our solution in this region?



One Possibility. Let

$$u_1(x,t) = \begin{cases} 0 & \text{for } x < \dfrac{t}{2} \\ 1 & \text{for } x > \dfrac{t}{2}. \end{cases}$$

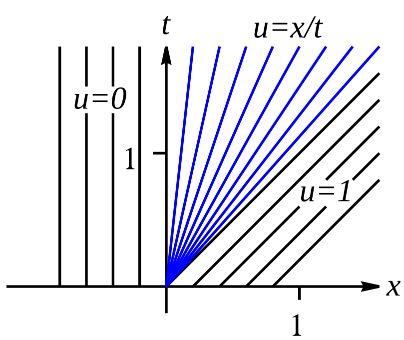
Clearly, $u_1(x,t)$ is a classical solution on either side of the curve of discontinuity $x = \frac{t}{2}$. In addition, from the work of the previous example, it is easy to see that $u_1(x,t)$ satisfies the Rankine-Hugoniot jump condition along the curve of discontinuity. Therefore, $u_1(x,t)$ is a weak solution of (3.16) satisfying (3.17). However, this is not the only possible solution.



Another Possibility. Let

$$u_2(x,t) = \begin{cases} 0 & \text{for } x \leq 0 \\ \dfrac{x}{t} & \text{for } 0 \leq x \leq t \\ 1 & \text{for } x \geq t. \end{cases}$$

Notice that $u_2(x,t)$ is a continuous solution of (3.16), (3.17). This type of solution which "fans" the wedge $0 < x < t$ is called a **rarefaction wave**.



We have found two different solutions. Is it possible, however, that one solution is more physically realistic? If so, we would like to consider that our "real" solution. Below we introduce the notion of an *entropy condition*. Solutions of quasilinear equations of the form (3.2) which satisfy this entropy condition are considered more physically realistic, and, thus, when looking for solutions of (3.2), we only allow for solutions which satisfy this extra condition. As we will see, solution $u_2(x,t)$ is considered to be the more physically realistic solution and we consider that our "real" solution.

$\diamond$

3.3 Entropy Condition

Consider a quasilinear equation of the form

$$u_t + [f(u)]_x = 0.$$

This equation can also be written in the form

$$u_t + f'(u)u_x = 0. \tag{3.18}$$

The characteristic equations associated with (3.18) are given by

$$\begin{aligned} \frac{dx}{ds} &= f'(z) \\ \frac{dt}{ds} &= 1 \\ \frac{dz}{ds} &= 0. \end{aligned}$$

From these equations, we see that the *speed* of a solution u is given by

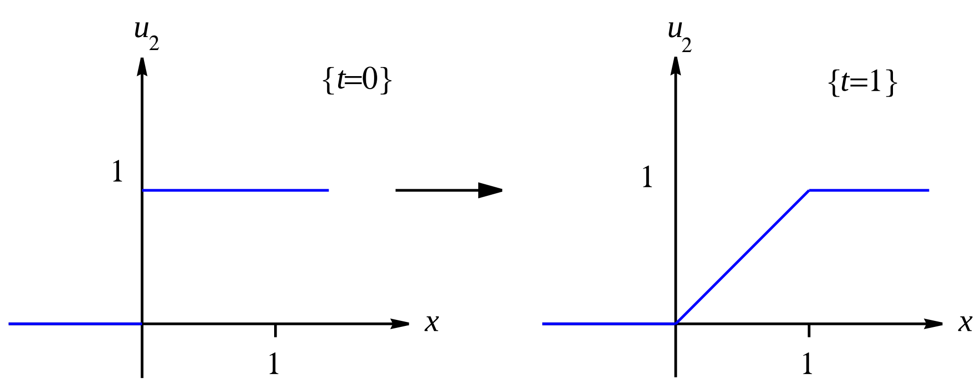
$$\frac{dx}{dt} = f'(u).$$

In particular, for Burgers' equation, the speed of a solution u is given by

$$\frac{dx}{dt} = u.$$

This says that taller waves move faster than shorter waves. In the case of Examples 5 and 6, the initial wave was taller on the left, and, thus moving faster than the wave on the right. As a result, we expected the part of the wave to the left to overtake the part of the wave to the right and cause the wave to break. This resulted in our curve of discontinuity.

In Example 7, however, for our initial data, the wave is higher to the right. Consequently, we expect the part of the wave to the right to move faster. Physically, therefore, we don't want to allow for solution $u_1(x,t)$. Instead, we accept $u_2(x,t)$ as a physically more realistic solution. See the movie of u_2 below.



Let's make these ideas more precise. In particular, for an equation of the form (3.18), we only allow for a curve of discontinuity in our solution $u(x,t)$ if the wave to the left is moving faster than the wave to the right. That is, we only allow for a curve of discontinuity between u^- and u^+ if

$$f'(u^-) > \sigma > f'(u^+). \tag{3.19}$$

This is known as the **entropy condition**. We say that a curve of discontinuity is a **shock curve** for a solution u if the curve satisfies the Rankine-Hugoniot jump condition and the entropy condition for that solution u. Therefore, to eliminate the physically less realistic solutions, we only "accept" solutions u for which curves of discontinuity in the solution are shock curves. We state this more precisely as follows.

Consider the initial-value problem,

$$\begin{cases} u_t + [f(u)]_x = 0 & x \in \mathbb{R}, t \geq 0 \\ u(x,0) = \phi(x). \end{cases} \tag{3.20}$$

We say u is a weak, **admissible** solution of (3.20) only if u is a weak solution such that any curve of discontinuity for u is a shock curve.

In Example 7, possibility one, for $u^- = 0$, $u^+ = 1$, $\sigma = \xi'(t) = \frac{1}{2}$,

$$f'(u^-) = u^- = 0 \not> \frac{1}{2} \not> 1 = u^+ = f'(u^+).$$

Therefore, u_1 does not satisfy the entropy condition along the curve of discontinuity $x = t/2$. Consequently, $x = t/2$ is not a shock curve, and, therefore, u_1 is not an admissible solution. Solution u_2, however, is a continuous solution. Therefore, we accept this solution as the physically relevant one.

We now turn to initial-value problems of the form (3.20) when f has a particular structure. We say a function f is **uniformly convex** if there exists a constant $\theta > 0$ such that $f'' \geq \theta > 0$. In particular, this means f' is strictly increasing. If f' is strictly increasing, then u will satisfy the entropy condition (3.19) if and only if

$$u^- > u^+$$

on any curve of discontinuity. That is to say, for f uniformly convex, u will be a weak, admissible solution to

$$u_t + [f(u)]_x = 0,$$

if and only if u satisfies the Rankine-Hugoniot condition (3.9) and u satisfies

$$u^- > u^+$$

along any curves of discontinuity.

3.4 Riemann's Problem

In this section, we study the initial value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases} \tag{3.21}$$

for the case when f is uniformly convex and the initial data is piecewise constant; that is,

$$\phi(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0. \end{cases} \tag{3.22}$$

The initial-value problem (3.21), (3.22) is known as **Riemann's Problem**.

Theorem 8. (See Evans, Chap. 3) *For f uniformly convex, there exists a unique weak, admissible* solution to Riemann's problem* (3.21), (3.22).

1. *If $u^- > u^+$, then the admissible solution has a shock curve of speed σ and the solution is given by*
$$u(x,t) = \begin{cases} u^- & \frac{x}{t} < \sigma \\ u^+ & \frac{x}{t} > \sigma \end{cases} \tag{3.23}$$
where $\sigma = [f(u)]/[u]$.

2. *If $u^- < u^+$, then the solution has a rarefaction wave and the solution is given by*
$$u(x,t) = \begin{cases} u^- & \frac{x}{t} < f'(u^-) \\ G(x/t) & f'(u^-) < \frac{x}{t} < f'(u^+) \\ u^+ & \frac{x}{t} > f'(u^+), \end{cases} \tag{3.24}$$
where $G \equiv (f')^{-1}$.

**Remark.* For technical reasons, Evans presents a more precise definition of *admissible* to prove uniqueness. We will omit the proof of uniqueness here. See Evans.

Proof. First, let's look at (1). Clearly, $u(x,t)$ defined in (3.23) is a classical solution of (3.21), (3.22) on either side of the curve $x = \sigma t$. In addition, the curve of discontinuity satisfies the Rankine-Hugoniot jump condition (3.9), so $u(x,t)$ is a weak solution. In addition, $u^- > u^+$ and f is uniformly convex. Therefore, u satisfies the entropy condition (3.19).

Now, we look at (2). For u defined by (3.24), u is a classical solution of (3.21), (3.22) to the left of $x = f'(u^-)t$ and to the right of $x = f'(u^+)t$. Now we check that u is a solution for $f'(u^-) < x/t < f'(u^+)$. By use of the chain rule, we have

$$\begin{aligned} u_t + [f(u)]_x &= G'(x/t) \cdot (-x/t^2) + f'(G(x/t)) \cdot [G(x/t)]_x \\ &= G'(x/t) \cdot (-x/t^2) + f'((f')^{-1}(x/t)) \cdot G'(x/t) \cdot (1/t) \\ &= G'(x/t) \cdot (-x/t^2) + G'(x/t) \cdot (x/t^2) = 0. \end{aligned}$$

Therefore, we have shown that u is a classical solution in each of the three regions in which it's defined. Moreover, along the curve $x/t = f'(u^-)$, $G(x/t) = (f')^{-1}(x/t) = u^-$ implies $u(x,t)$ is continuous across the curve $x/t = f'(u^-)$. Similarly, $u(x,t)$ defined in (3.24) is continuous across the curve $x/t = f'(u^+)$. Therefore, u has no curves of discontinuity and thus satisfies the entropy condition. It is easy to check that u satisfies (3.3), and, is therefore a weak solution of (3.21), (3.22).

□

Example 9. Consider the problem

$$\begin{cases} u_t + uu_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The characteristic equations are given by

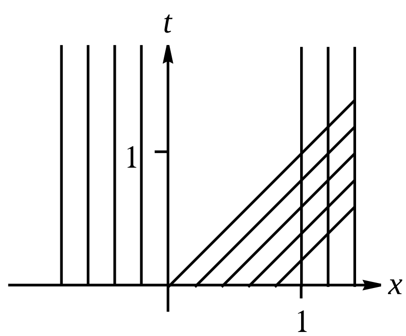
$$\frac{dt}{ds} = 1$$
$$\frac{dx}{ds} = z$$
$$\frac{dz}{ds} = 0$$

with initial conditions

$$t(r,0) = 0$$
$$x(r,0) = r$$
$$z(r,0) = \phi(r).$$

We know the solution is constant along the projected characteristic curves given by $x(r,t) = \phi(r)t + r$.

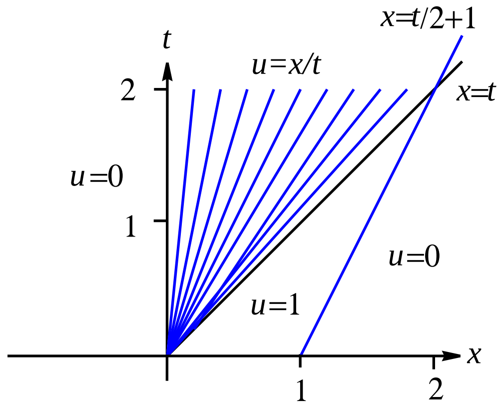
If $r < 0$, then $\phi(r) = 0$ implies $x = r$ and $u = 0$ along these curves. If $0 < r < 1$, then $\phi(r) = 1$ implies $x = t + r$ and $u = 1$ along these curves. If $r > 1$, then $\phi(r) = 0$ implies $x = r$ and $u = 0$ along these curves.



Therefore, we get a rarefaction wave between $x = 0$ and $x = t$ and a shock is formed by the intersection of the lines $x = t + r$ for $0 < r < 1$ and $x = 1$. By the Rankine-Hugoniot jump condition, the speed σ of the shock must satisfy

$$\begin{aligned} \sigma &= \frac{[f(u)]}{[u]} \\ &= \frac{\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2}}{u^- - u^+} \\ &= \frac{1/2 - 0}{1 - 0} = \frac{1}{2}. \end{aligned}$$

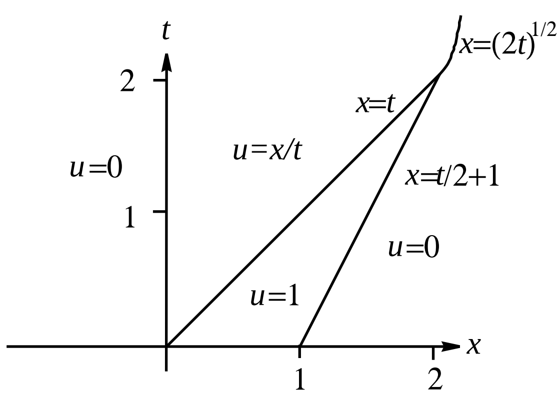
Therefore, the shock curve is given by $x - 1 = \frac{1}{2}t$ or $x = 1 + \frac{t}{2}$. So, our weak solution u will take on the values shown below.



Notice at $t = 2$, however, the rarefaction wave hits the shock curve $x = 1 + \frac{t}{2}$. We need the jump across the shock to satisfy the Rankine-Hugoniot jump condition. To the left of the jump, we have $u^- = x/t$ and to the right, we have $u^+ = 0$. The Rankine-Hugoniot jump condition becomes

$$\begin{aligned}
\sigma &= \frac{[f(u)]}{[u]} \\
&= \frac{\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2}}{u^- - u^+} \\
&= \frac{\frac{1}{2}\left(\frac{x}{t}\right)^2 - 0}{\frac{x}{t} - 0} \\
&= \frac{x}{2t}.
\end{aligned}$$

Therefore, we get a new shock curve emanating from the point $(2, 2)$ with speed $\sigma = \xi'(t) = x/2t$. This curve is given by $x(t) = \sqrt{2t}$. Therefore the weak solution u will take on the values shown below.



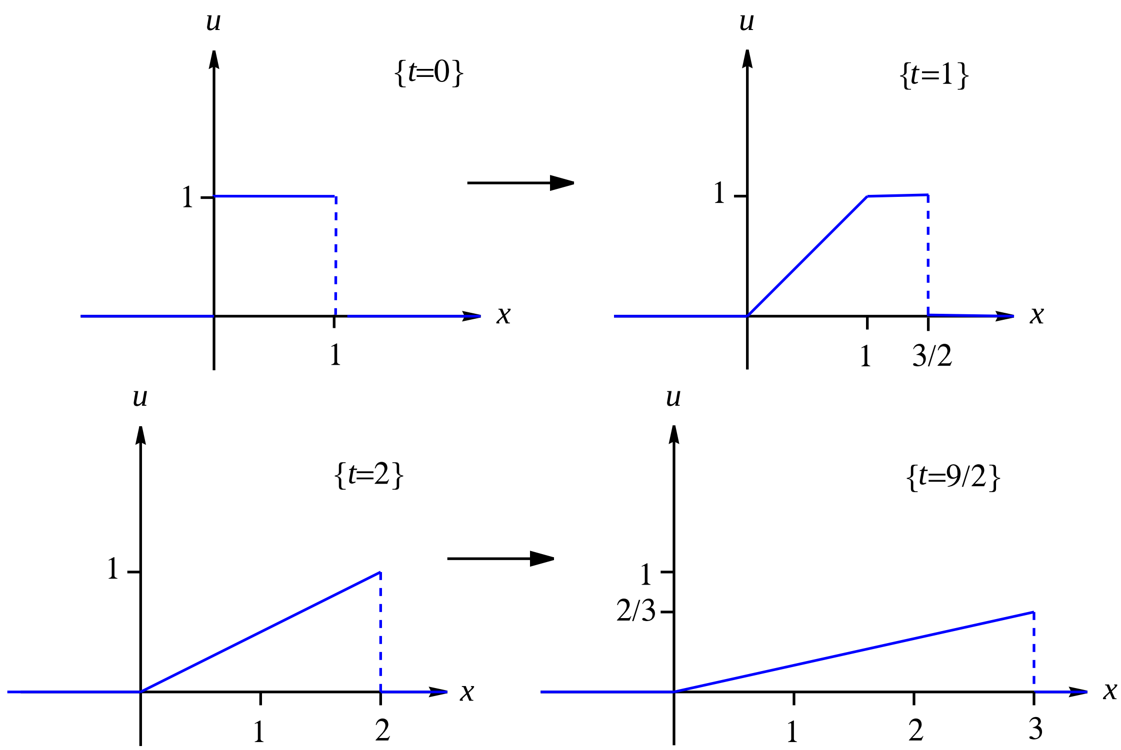
To summarize, we have

$$u(x,t) = \begin{cases} 0 & \text{for } x < 0 \\ x/t & \text{for } 0 < x < t \\ 1 & \text{for } t < x < 1 + \dfrac{t}{2} \\ 0 & \text{for } x > 1 + \dfrac{t}{2} \end{cases}$$

for $t \leq 2$, and

$$u(x,t) = \begin{cases} 0 & \text{if } x < 0 \\ x/t & \text{if } 0 < x < \sqrt{2t} \\ 0 & \text{if } \sqrt{2t} < x \end{cases}$$

for $t \geq 2$.



◇

3.5 Long-Time Asymptotics

Notice in Example 9 that $|u| \to 0$ as $t \to +\infty$. In fact, by using the solution formula, we see that for $0 \leq t \leq 2$, $|u(x,t)| \leq 1$ and for $t \geq 2$,

$$|u(x,t)| \leq \frac{x}{t} \leq \frac{\sqrt{2t}}{t} = \frac{\sqrt{2}}{\sqrt{t}}.$$

In summary, we have

$$|u(x,t)| \leq \frac{\sqrt{2}}{\sqrt{t}}$$

for all $t > 0$, all $x \in \mathbb{R}$. Thus, u decays to zero like $1/\sqrt{t}$ as $t \to +\infty$. This property is true in general for solutions of the initial-value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases} \tag{3.25}$$

assuming

1. $\phi(x)$ is bounded and integrable
2. f is smooth, uniformly convex and $f(0) = 0$.

We state this more precisely in the following theorem.

Theorem 10. (See Evans, Chapter 3) *Consider an initial-value problem of the form* (3.25) *where ϕ and f satisfy the hypothesis indicated. Then the solution u of* (3.25) *satisfies the following decay estimate. There exists a constant $C > 0$ such that*

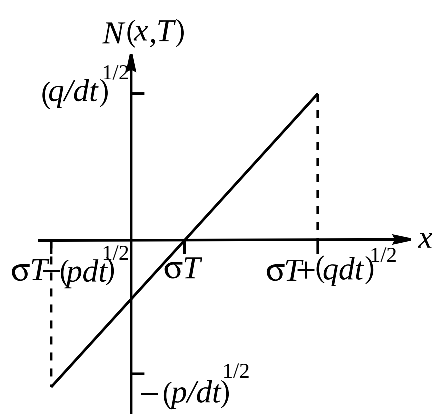
$$|u(x,t)| \leq \frac{C}{\sqrt{t}}$$

for all $t > 0$, all $x \in \mathbb{R}$.

Proof. See Evans. □

In addition to the solution of (3.25) decaying to zero as $t \to +\infty$, we can say more about the profile of the solution. In general, solutions of (3.25) decay to an *N-wave*. For constants p, q, d, σ we define an **N-wave** to be a function of the form

$$N(x,t) = \begin{cases} \frac{1}{d}\left(\frac{x}{t} - \sigma\right) & \text{for } -(pdt)^{1/2} < x - \sigma t < (qdt)^{1/2} \\ 0 & \text{otherwise.} \end{cases} \tag{3.26}$$



We now define a specific N-wave using the constants defined below. Let p, q, d and σ be defined as follows. Let

$$\begin{aligned} p &= -2 \min_{y\in\mathbb{R}} \int_{-\infty}^{y} \phi(x)\,dx \\ q &= 2 \max_{y\in\mathbb{R}} \int_{y}^{\infty} \phi(x)\,dx \\ d &= f''(0) \\ \sigma &= f'(0). \end{aligned} \tag{3.27}$$

Theorem 11. (See Evans, Chapter 3) *For an initial-value problem of the form* (3.25) *where f is smooth and uniformly convex and ϕ has compact support, then for p, q, d, σ defined in* (3.27) *and $N(x,t)$ defined in* (3.26) *for this choice of constants p, q, d, σ, there exists a constant $C > 0$ such that the solution u of* (3.25) *satisfies*

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)|\,dx \leq \frac{C}{\sqrt{t}}$$

for all $t > 0$.

Proof. See Evans. □

Example 12. For Example 9, $p = 0$, $q = 2$, $f(u) = u^2/2$ implies $\sigma = f'(0) = 0$ and $d = f''(0) = 1$. Therefore,

$$N(x,t) = \begin{cases} x/t & \text{for } 0 < x < \sqrt{2t} \\ 0 & \text{otherwise,} \end{cases}$$

and Theorem 11 says there exists a constant $C > 0$ such that

$$\int_{-\infty}^{\infty} |u(x,t) - N(x,t)|\,dx \leq \frac{C}{\sqrt{t}}.$$

We see that this statement is true because for $0 \leq t \leq 2$,

$$\begin{aligned} \int_{-\infty}^{\infty} |u(x,t) - N(x,t)|\,dx &= \int_{0}^{1+t/2} |u(x,t) - N(x,t)|\,dx \\ &= \int_0^t |x/t - x/t|\,dx + \int_t^{\sqrt{2t}} |1 - x/t|\,dx + \int_{\sqrt{2t}}^{1+t/2} |1 - 0|\,dx \\ &\leq (1 + \sqrt{2t}/t)[\sqrt{2t} - t] + [1 + t/2 - \sqrt{2t}] \\ &= 3 - \frac{t}{2} - \sqrt{2t} \leq \frac{6\sqrt{2}}{\sqrt{t}}, \end{aligned}$$

while $u(x,t) = N(x,t)$ for $t \geq 2$. ◇

3.6 Oleinik Entropy Condition.

In this section, we consider quasilinear first-order initial-value problems in the case when f is not uniformly convex.

Recall for an equation of the form

$$u_t + [f(u)]_x = 0, \tag{3.28}$$

$f'(u)$ is the speed of the wave with height u. Recall the entropy condition

$$f'(u^-) > \sigma > f'(u^+) \tag{3.29}$$

where $\sigma = [f(u)]/[u]$. For the case when f is uniformly convex, requiring that curves of discontinuity satisfy the entropy condition (3.29) guarantees uniqueness of solutions. In the case, when f is not uniformly convex, condition (3.29) is not enough to guarantee uniqueness of solutions.

First, we make some remarks on the entropy condition. In general, we only want to allow for a curve of discontinuity $x = \xi(t)$ if the speed of the wave to the left of the curve is greater than *or equal to* the speed $\sigma = \xi'(t)$ of the curve of discontinuity, and σ is greater than or equal to the speed of the wave to the right of the curve of discontinuity. If f is uniformly convex, meaning f' is strictly increasing, then for $u^- \neq u^+$, there are two possibilities:

1. $f'(u^-) > \sigma > f'(u^+)$
2. $f'(u^-) < \sigma < f'(u^+)$.

Therefore, for f uniformly convex, we only want to allow for a curve of discontinuity in the case when

$$f'(u^-) > \sigma > f'(u^+).$$

Consequently, we take this as our entropy condition.

More generally, for f not necessarily uniformly convex, we only want to allow for a curve of discontinuity $x = \xi(t)$ in the case when

$$f'(u^-) \geq \sigma \geq f'(u^+).$$

As shown in the following example, however, this condition is not enough to guarantee uniqueness of solutions.

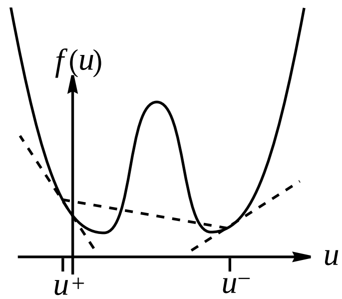
Example 13. Consider the initial-value problem

$$\begin{aligned} &u_t + [f(u)]_x = 0, \qquad t \geq 0 \\ &u(x,0) = \phi(x) \end{aligned}$$

where

$$\phi(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0 \end{cases}$$

for f, u^-, u^+ shown below.



Let

$$u(x,t) = \begin{cases} u^- & x/t < \sigma \\ u^+ & x/t > \sigma \end{cases}$$

where $\sigma \equiv [f(u)]/[u]$. By the definition of the curve of discontinuity $x = \sigma t$, the Rankine-Hugoniot jump condition is satisfied. As can be seen in the figure above, the entropy condition is satisfied as $f'(u^-) > \sigma > f'(u^+)$.

We claim that there exists another solution which satisfies the condition

$$f'(u^-) \geq \sigma \geq f'(u^+)$$

along any curve of discontinuity. In particular, see Example 16 below. Defining u_2, u_3, u as stated there, we see that the Rankine-Hugoniot jump condition is satisfied along both curves of discontinuity, and, in addition, the condition

$$f'(u^-) \geq \sigma \geq f'(u^+)$$

is satisfied along the curves of discontinuity $x = \xi_1(t)$, $x = \xi_2(t)$. Consequently, we have found two weak solutions which satisfy the condition

$$f'(u^-) \geq \sigma \geq f'(u^+)$$

along any curves of discontinuity. In order to guarantee uniqueness of solutions, we introduce the *Oleinik entropy condition.*

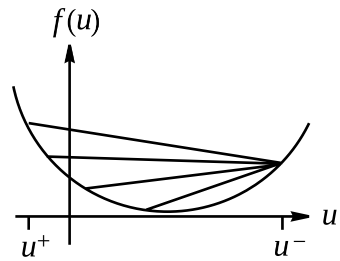
$\diamond$

So, as shown in the previous example, in the case when f is *not* uniformly convex, the entropy condition is not enough to guarantee uniqueness of solutions. Here we introduce a stronger condition which will guarantee uniqueness of solutions of (3.28) in the case when f is not uniformly convex.

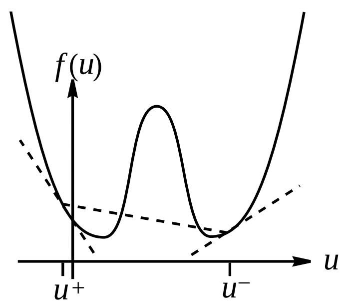
Recall that for the case when f is uniformly convex, the entropy condition (3.29) is satisfied if and only if $u^- > u^+$ along the curve of discontinuity. In addition, we see that for f uniformly convex, condition (3.29) is equivalent to the condition

$$\frac{f(u^-) - f(u)}{u^- - u} \geq \frac{f(u^-) - f(u^+)}{u^- - u^+} \tag{3.30}$$

for all u such that $u^- > u > u^+$.



In the case when f is not uniformly convex, however, (3.29) and (3.30) are not equivalent. For example, consider f, u^-, u^+ shown below. Then (3.29) holds, but (3.30) does not hold.



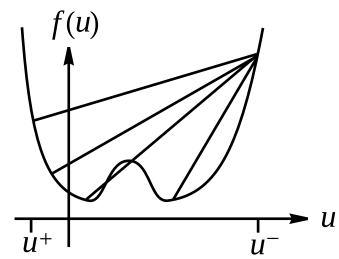
However, if (3.30) is satisfied, then (3.29) is satisfied. In this sense, (3.30) is a stronger statement. The condition that

$$\frac{f(u^-) - f(u)}{u^- - u} \geq \frac{f(u^-) - f(u^+)}{u^- - u^+} \tag{3.31}$$

for all u between u^- and u^+ is known as the **Oleinik entropy condition**. *Note: u^- does not need to be greater than u^+ for* (3.31) *to hold.* (See examples below.)

The Oleinik entropy condition guarantees uniqueness of solutions of (3.28) in the case when f is not uniformly convex. We say a **shock is admissible** if it satisfies the Oleinik entropy condition. We say a weak, admissible solution u of (3.28) is an **admissible entropy** solution if all shocks are admissible.

Example 14. Consider the case when $u^- > u^+$ and $f(u)$ is shown below. By drawing chords connecting $(u^-, f(u^-))$ with all points $(u, f(u))$ for $u^+ \leq u \leq u^-$, we can see that the Oleinik entropy condition (3.31) is satisfied.



Therefore, for the initial-value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

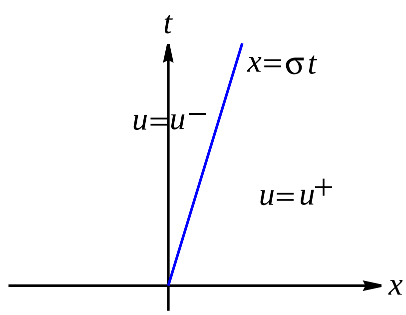
where

$$\phi(x) = \begin{cases} u^- & \text{for } x < 0 \\ u^+ & \text{for } x > 0 \end{cases}$$

with $f(u)$ as shown above, a shock is admissible, and the solution is given by

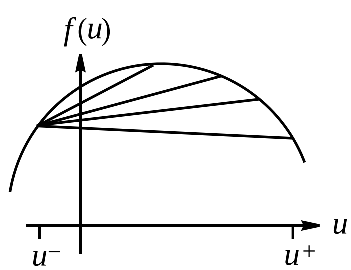
$$u(x,t) = \begin{cases} u^- & \text{for } x/t < \sigma \\ u^+ & \text{for } x/t > \sigma \end{cases}$$

where as usual $\sigma = [f(u)]/[u]$.



◇

Example 15. Now consider the case when $u^+ > u^-$ and $f(u)$ is as shown below. Again, by drawing chords connecting $(u^-, f(u^-))$ and $(u, f(u))$ for all u between u^- and u^+, we see that the Oleinik entropy condition is satisfied.



Therefore, a shock is admissible and the solution of the initial-value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} u^- & \text{for } x < 0 \\ u^+ & \text{for } x > 0 \end{cases}$$

with $f(u)$ as shown above is given by

$$u(x,t) = \begin{cases} u^- & \text{for } x/t < \sigma \\ u^+ & \text{for } x/t > \sigma. \end{cases}$$

◇

To summarize, we have shown the following. If either of the following hold:

1. $u^- > u^+$ and the chord connecting $(u^-, f(u^-))$ and $(u^+, f(u^+))$ lies above the graph of $f(u)$
2. $u^+ > u^-$ and the chord connecting $(u^-, f(u^-))$ and $(u^-, f(u^+))$ lies below the graph of $f(u)$,

then a shock is admissible and the solution of

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} u^- & \text{for } x < 0 \\ u^+ & \text{for } x > 0, \end{cases}$$

is given by

$$u(x,t) = \begin{cases} u^- & \text{for } x/t < \sigma \\ u^+ & \text{for } x/t > \sigma \end{cases}$$

where $\sigma = [f(u)]/[u]$ is defined by the Rankine-Hugoniot jump condition (3.9).

Now what happens if u^-, u^+ and f are such that neither of the conditions above hold? We will get a combination of rarefaction waves and shock curves.

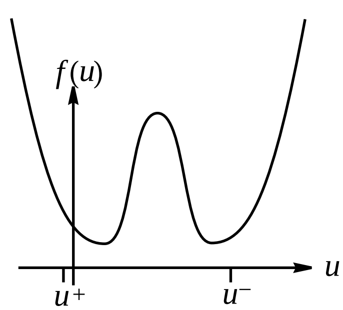
Example 16. Consider the initial-value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases} \tag{3.32}$$

where

$$\phi(x) = \begin{cases} u^- & \text{for } x < 0 \\ u^+ & \text{for } x > 0 \end{cases} \tag{3.33}$$

with f, u^- and u^+ as shown in the picture below.



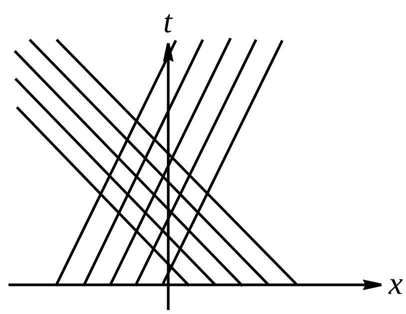
We see that the characteristics for this equation are given by

$$\frac{dx}{ds} = f'(z)$$
$$\frac{dt}{ds} = 1$$
$$\frac{dz}{ds} = 0$$

with initial conditions

$$x(r,0) = r$$
$$t(r,0) = 0$$
$$z(r,0) = \phi(r)$$

which implies the projected characteristics are given by $x(t) = f'(\phi(r))t + r$. For $r < 0$, $\phi(r) = u^-$ and we see from the picture above that $f'(u^-) > 0$. For $r > 0$, $\phi(r) = u^+$ and we see from the picture above that $f'(u^+) < 0$. Therefore, our projected characteristics look as shown in the figure below.



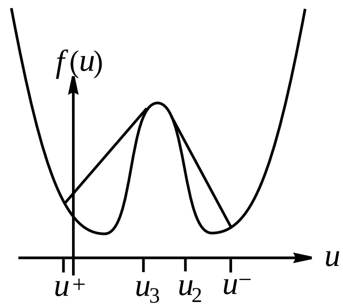
One possibility for a weak solution of (3.32), (3.33) would be to put in a shock curve between u^- and u^+. Of course, this shock curve would need to satisfy the Rankine-Hugoniot jump condition $\sigma = [f(u)]/[u]$, and, therefore, our weak solution would be given by

$$u(x,t) = \begin{cases} u^- & \text{for } x/t < \sigma \\ u^+ & \text{for } x/t > \sigma \end{cases}$$

where $\sigma = [f(u)]/[u]$. However, this solution does not satisfy the Oleinik entropy condition (3.31) along the curve of discontinuity $x/t = \sigma$. Therefore, this is not the admissible entropy solution.

So, how do we find the admissible entropy solution? We use the following "rubberband" method. First find the value $u_2 < u^-$ which is closest to u^- such that

$$f'(u_2) = \frac{f(u^-) - f(u_2)}{u^- - u_2}.$$



Then for $u_2 < u < u^-$,

$$\frac{f(u^-) - f(u)}{u^- - u} \geq \frac{f(u^-) - f(u_2)}{u^- - u_2}.$$

Therefore, u^-, u_2 satisfy the Oleinik entropy condition, so we can put in a shock curve, $x = \xi_1(t)$ from u^- to u_2. This shock curve must satisfy the Rankine-Hugoniot jump condition (3.9). Therefore,

$$\xi_1'(t) = \frac{f(u^-) - f(u_2)}{u^- - u_2} = f'(u_2).$$

Next, find u_3 such that

$$f'(u_3) = \frac{f(u_3) - f(u^+)}{u_3 - u^+}.$$

Therefore,

$$\frac{f(u_3) - f(u)}{u_3 - u} \geq \frac{f(u_3) - f(u^+)}{u_3 - u^+}$$

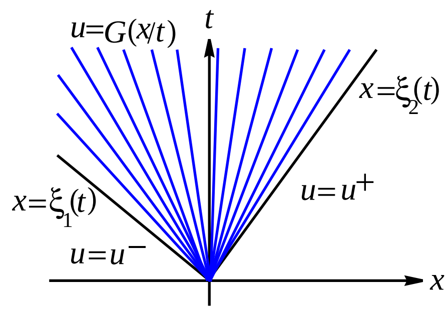
for all u between u^+ and u_3, so u_3, u^+ satisfy the Oleinik entropy condition and we can put in a shock curve $x = \xi_2(t)$ between u_3 and u^+. This shock curve will satisfy

$$\xi_2'(t) = \frac{f(u_3) - f(u^+)}{u_3 - u^+} = f'(u_3).$$

Then between these shock curves, we put in a rarefaction wave from u_2 to u_3. Therefore, our solution is given by

$$u(x,t) = \begin{cases} u^- & \text{for } x/t < f'(u_2) \\ G(x/t) & \text{for } f'(u_2) < x/t < f'(u_3) \\ u^+ & \text{for } f'(u_3) < x/t \end{cases}$$

where $G = (f')^{-1}$.



$\diamond$

For further information on the Oleinik entropy condition and weak solutions of the initial-value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

in the case when f does not have any convexity assumptions, see the following reference.

Reference: D. Ballou, *Solutions to Nonlinear Hyperbolic Cauchy Problems without Convexity Conditions*, Transactions of the American Mathematical Society, **152**, Dec. 1970, 441-460.