Stanford Mathematics Department
Math 205A Lecture Supplement #4
Borel Regular & Radon Measures

1 Borel Regular Measures

Recall that a Borel measure on a topological space X is a measure defined on the collection of Borel sets, and an outer measure μ on X is said to be a *Borel-regular outer measure* if all Borel sets are μ-measurable and if for each subset $A \subset X$ there is a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$. (Notice that this does *not* imply $\mu(B \setminus A) = 0$ unless A is μ-measurable and $\mu(A) < \infty$.) Also if ν is a Borel measure on X, then we get a Borel regular outer measure μ on X by defining $\mu(Y) = \inf \nu(B)$ where the inf is taken over all Borel sets B with $B \supset Y$, and this outer measure μ coincides with ν on all the Borel sets. (See Q.1 of hw2.)

Also, if μ is a Borel regular outer measure on X and if $A \subset X$ is μ-measurable with $\mu(A) < \infty$, then we claim $\mu \llcorner A$ is also Borel regular. Here $\mu \llcorner A$ is the outer measure on X defined by

$$(\mu \llcorner A)(Y) = \mu(A \cap Y).$$

To check this claim first observe that if E is μ-measurable and $Y \subset X$ is arbitrary then $(\mu \llcorner A)(Y) = \mu(A \cap Y) = \mu(A \cap Y \cap E) + \mu(A \cap Y \setminus E) = (\mu \llcorner A)(Y \cap E) + (\mu \llcorner A)(Y \setminus E)$, hence E is also $(\mu \llcorner A)$-measurable. In particular all Borel sets are $(\mu \llcorner A)$-measurable, so it remains to prove that for each $Y \subset X$ there is a Borel set $B \supset Y$ with $(\mu \llcorner A)(B) = (\mu \llcorner A)(Y)$. To prove this, first use the Borel regularity of μ and the fact that A is measurable of finite measure to pick Borel sets $B_1 \supset Y \cap A$ and $B_2 \supset A$ with $\mu(B_1) = \mu(Y \cap A)$ and $\mu(B_2 \setminus A) = 0$, and then pick a Borel set $B_3 \supset B_2 \setminus A$ with $\mu(B_3) = 0$. Then $Y \subset B_1 \cup (X \setminus A) \subset B_1 \cup (X \setminus B_2) \cup B_3$ (which is a Borel set) and $(\mu \llcorner A)(Y) \le (\mu \llcorner A)(B_1 \cup (X \setminus B_2) \cup B_3) = \mu((A \cap B_1) \cup (A \cap B_3)) \le \mu(B_1) = \mu(Y \cap A) = (\mu \llcorner A)(Y)$.

We now state and prove an important regularity property of Borel regular outer measures:

1.1 Theorem. *Suppose X is a topological space with the property that every closed subset of X is the countable intersection of open sets (this trivially holds e.g. if X is a metric space), suppose μ is a Borel-regular outer measure on X, and suppose that $X = \cup_{j=1}^{\infty} V_j$, where $\mu(V_j) < \infty$ and V_j is open for each $j = 1, 2, \ldots$. Then*

$$\mu(A) = \inf_{U \text{ open}, U \supset A} \mu(U) \tag{1}$$

for each subset $A \subset X$, and

$$\mu(A) = \sup_{C \text{ closed}, C \subset A} \mu(C) \tag{2}$$

for each μ-measurable subset $A \subset X$.

1.2 Remark: In case X is a locally compact separable metric space (thus for each $x \in X$ there is $\rho > 0$ such that the closed ball $B_\rho(x) = \{y \in X : d(x, y) \le \rho\}$ is compact, and X has a countable dense subset), the condition $X = \cup_{j=1}^{\infty} V_j$ with V_j open and $\mu(V_j) < \infty$ is *automatically satisfied provided $\mu(K) < \infty$ for each compact K*. Furthermore in this case we have from (2) above that

$$\mu(A) = \sup_{K \text{ compact}, K \subset A} \mu(K)$$

for each μ-measurable subset $A \subset K$ with $\mu(A) < \infty$, because under the above conditions on X any closed set C can be written $C = \cup_{i=1}^{\infty} K_i$, K_i compact.

Proof of 1.1. We assume first that $\mu(X) < \infty$. By Borel regularity of μ, for any given $A \subset X$ we can select a Borel set $B \supset A$ with $\mu(B) = \mu(A)$, so it clearly suffices to check (1) in the special case when A is a Borel set. Now let

$$\mathcal{A} = \{ \text{ Borel sets } A : (1) \text{ holds}\}.$$

Trivially $\mathcal{A}$ contains all open sets and one readily checks that A is closed under both countable unions and intersections, as follows:

If $A_1, A_2, \ldots \in \mathcal{A}$ then for any given $\varepsilon > 0$ there are open $U_1, U_2, \ldots$ with $U_j \supset A_j$ and $\mu(U_j \setminus A_j) \le 2^{-j}\varepsilon$. Now one easily checks $\cup_j U_j \setminus (\cup_k A_k) \subset \cup_j (U_j \setminus A_j)$ and $\cap_j U_j \setminus (\cap_k A_k) \subset \cup_j (U_j \setminus A_j)$ so by subadditivity we have $\mu(\cup_{j=1}^{\infty} U_j \setminus (\cup_k A_k)) < \varepsilon$ and $\lim_{N \to \infty} \mu(\cap_{j=1}^{N} U_j \setminus (\cap_k A_k)) = \mu(\cap_{j=1}^{\infty} U_j \setminus (\cap_k A_k)) < \varepsilon$, so both $\cup_k A_k$ and $\cap_k A_k$ are in $\mathcal{A}$ as claimed.

In particular $\mathcal{A}$ must also contain the *closed sets*, because any closed set in X can be written as a countable intersection of open sets and all the open sets are in $\mathcal{A}$. Thus if we let $\widetilde{\mathcal{A}} = \{A \in \mathcal{A} : X \setminus A \in \mathcal{A}\}$ then $\widetilde{\mathcal{A}}$ is a σ-algebra containing all the closed sets, and hence $\widetilde{\mathcal{A}}$ contains all the Borel sets. Thus $\mathcal{A}$ contains all the Borel sets and (1) is proved in case $\mu(X) < \infty$.

To check (2) in case $\mu(X) < \infty$ we can just apply (1) to $X \setminus A$: thus for each $i = 1, 2, \dots$ there is an open set $U_i \supset X \setminus A$ with $\mu(X) - \mu(A) + 1/i = \mu(X \setminus A) + 1/i > \mu(U_i) = \mu(X) - \mu(C_i)$, where $C_i = X \setminus U_i \subset X \setminus (X \setminus A) = A$. Thus $C_i \subset A$ is closed and $\mu(C_i) > \mu(A) - 1/i$.

In case $\mu(X) = \infty$, to prove (1) we first take a Borel set B with $\mu(B) = \mu(A)$ and then apply the above result for finite measures to the Borel regular measure $\mu \llcorner V_j$, $j = 1, 2, \dots$, thus obtaining, for given $\varepsilon > 0$, open $U_j \supset B$ with $(\mu \llcorner V_j)(U_j \setminus B) < \varepsilon 2^{-j}$ and hence $\mu(U_j \cap V_j \setminus B) \le \varepsilon 2^j$ and by subadditivity this gives

$$\mu(\cup(U_j \cap V_j) \setminus B) < \varepsilon.$$

Since $\cup(U_j \cap V_j)$ is an open set containing B (hence A), this is the required result.

Similarly to check (2) in case $\mu(X) = \infty$, we apply (2) for the finite measure case to the Borel regular measure $\mu \llcorner V_j$, giving closed $C_j \subset X$ such that $C_j \subset A$ and $(\mu \llcorner V_j)(A \setminus C_j) < \varepsilon 2^{-j}$ for each $j = 1, 2, \dots$, and hence $\mu(A \cap V_j \setminus \cup_{k=1}^\infty C_k) < \varepsilon 2^{-j}$, so by subadditivity and the fact that $\cup V_j = X$ we have $\mu(A \setminus (\cup_{k=1}^\infty C_k)) < \varepsilon$ and hence $\mu(A) \le \mu(\cup_{k=1}^\infty C_k) + \varepsilon = \lim_{N \to \infty} \mu(\cup_{k=1}^N C_k) + \varepsilon$. Since $\cup_{k=1}^N C_k$ is closed for each N this is the required result. □

Using the above lemma we now prove Lusin's Theorem:

1.3 Theorem (Lusin's Theorem.) *Let μ be a Borel regular outer measure on a topological space X having the property that every closed set can be expressed as the countable intersection of open sets (e.g. X is a metric space), let A be μ-measurable with $\mu(A) < \infty$, and let $f : A \to \mathbb{R}$ be μ-measurable. Then for each $\varepsilon > 0$ there is a closed set $C \subset X$ with $C \subset A$, $\mu(A \setminus C) < \varepsilon$, and $f|C$ continuous.*

Proof: For each $i = 1, 2, \dots$ and $j = 0, \pm 1, \pm 2, \dots$ let

$$A_{ij} = f^{-1}((j-1)/i, j/i],$$

so that A_{ij}, $j = 1, 2, \dots$, are p.w.d. sets in $\mathcal{A}$ and $\cup_{j=-\infty}^{\infty} A_{ij} = A$.

By the remarks preceding Theorem 1.1 we know that $\mu \llcorner A$ is a Borel regular outer measure and since it is finite we can apply Theorem 1.1 to it, and hence for given $\varepsilon > 0$ there are closed sets C_{ij} in X with $C_{ij} \subset A_{ij}$ with $(\mu \llcorner A)(A_{ij} \setminus C_{ij}) = \mu(A_{ij} \setminus C_{ij}) < 2^{-i-|j|-1}\varepsilon$, hence $\mu(A_{ij} \setminus (\cup_{\ell=-\infty}^{\infty} C_{i\ell})) < 2^{-i-|j|-1}\varepsilon$ and

hence

$$\mu(A \setminus (\cup_{\ell=-\infty}^{\infty} C_{i\ell})) < 2^{-i}\varepsilon$$

so for each $i = 1, 2, \dots$ there is a positive integer $J(i)$ such that

$$\mu(A \setminus (\cup_{|j| \le J(i)} C_{ij})) < 2^{-i}\varepsilon$$

Since $A \setminus (\cap_{i=1}^\infty (\cup_{|j| \le J(i)} C_{ij})) = \cup_{i=1}^\infty (A \setminus (\cup_{|j| \le J(i)} C_{ij}))$ (by De Morgan), this implies

$$\mu(A \setminus C) < \varepsilon,$$

where $C = \cap_{i=1}^\infty (\cup_{|j| \le J(i)} C_{ij})$ is a closed subset of A.

Now define $g_i : \cup_{|j| \le J(i)} C_{ij} \to \mathbb{R}$ by setting $g_i(x) \equiv (j-1)/i$ on C_{ij}, $|j| \le J(i)$. Then g_i is clearly continuous and its restriction to C is continuous for each i; furthermore by construction $0 \le f(x) - g_i(x) \le 1/i$ for each $x \in C$ and each $i = 1, 2, \dots$, so that $g_i|C$ converges uniformly to $f|C$ on C and hence $f|C$ is continuous. □

2 Radon Measures, Representation Theorem

In this section we work mainly in locally compact Hausdorff spaces, and for the reader's convenience we recall some basic definitions and preliminary topological results for such spaces.

Recall that a topological space is said to be Hausdorff if it has the property that for every pair of distinct points $x, y \in X$ there are open sets U, V with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In such a space *all compact sets are automatically closed*, the proof of which is as follows: observe that if $x \notin K$ then for each $y \in K$ we can (by definition of Hausdorff space) pick open U_y, V_y with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. By compactness of K there is a finite set $y_1, \dots, y_N \in K$ with $K \subset \cup_{j=1}^N V_{y_j}$. But then $\cap_{j=1}^N U_{y_j}$ is an open set containing x which is disjoint from $\cup_j V_{y_j}$ and hence disjoint from K, so that K is closed as claimed. In fact we proved a bit more: that for each $x \notin K$ there are disjoint open sets U, V with $x \in U$ and $K \subset V$. Then if L is another compact set disjoint from K we can repeat this for each $x \in L$ thus obtaining disjoint open U_x, V_x with $x \in U_x$ and $K \subset V_x$, and then compactness of L implies $\exists x_1, \dots, x_M \in L$ such that $L \subset \cup_{j=1}^M U_{x_j}$ and then $\cup_{j=1}^M U_{x_j}$ and $\cap_{j=1}^M V_{x_j}$ are disjoint open sets containing L and K respectively. By a simple inductive argument (left as an exercise) we can extend this to finite pairwise disjoint unions of compact subsets:

2.1 Lemma. *Let X be a Hausdorff space and $K_1, \dots, K_N$ be pairwise disjoint compact subsets of X. Then there are pairwise disjoint open subsets $U_1, \dots, U_N$*

with $K_j \subset U_j$ for each $j = 1, \ldots, N$.

Notice in particular that we have the following corollary of Lemma 2.1:

2.2 Corollary. *A compact Hausdorff space is normal: i.e. given closed disjoint subsets K_1, K_2 of a compact Hausdorff space, we can find disjoint open U_1, U_2 with $K_j \subset U_j$ for $j = 1, 2$.*

Most of the rest of the discussion here takes place in locally compact Hausdorff space: A space X is said to be *locally compact* if for each $x \in X$ there is a neighborhood U_x of x such that the closure $\overline{U}_x$ of U_x is compact.

An important preliminary lemma in such spaces is:

2.3 Lemma. *If X is a locally compact Hausdorff space and V is a nhd. of a point x, then there is a nhd. U_x of x such that $\overline{U}_x$ is a compact subset of V.*

Proof: First pick a neighborhood U_0 of x such that $\overline{U}_0$ is compact and define $W = U_0 \cap V$. Then $\overline{W}$ is compact and hence so is the closed subset $\overline{W} \setminus W$. Then $\overline{W} \setminus W$ and $\{x\}$ are disjoint compact sets so by Lemma 2.1 we can find disjoint open U_1, U_2 with $x \in U_1$ and $\overline{W} \setminus W \subset U_2$. Without loss of generality we can assume $U_1 \subset W$ (otherwise replace U_1 by $U_1 \cap W$). Then $\overline{U}_1 \subset X \setminus U_2 \subset X \setminus (\overline{W} \setminus W)$ and hence $\overline{U}_1 \subset W$. Thus the lemma is proved with $U_x = U_1$. □

Remark: In locally compact Hausdorff space, using Lemmas 2.1 and 2.3 it is easy to check that we can select the U_j in Lemma 2.1 above to have compact pairwise disjoint closures.

The following lemma is a version of the Urysohn lemma valid in locally compact Hausdorff space:

2.4 Lemma. *Let X be a locally compact Hausdorff space, $K \subset X$ compact, and $K \subset W$, W open. Then there is an open $V \supset K$ with $\overline{V} \subset W$, $\overline{V}$ compact, and an $f : X \to [0, 1]$ with $f \equiv 1$ in a neighborhood of K and* $\operatorname{spt} f \subset V$.

Proof: By Lemma 2.3 each $x \in K$ has a neighborhood U_x with $\overline{U}_x \subset W$ and $\overline{U}_x$ compact. Then by compactness of K we have $K \subset V \equiv \cup_{j=1}^N U_{x_j}$ for some finite collection $x_1, \ldots, x_N \in K$ and $\overline{V} = \cup_{j=1}^N \overline{U}_{x_j} \subset W$. Now $\overline{V}$ is compact, so by Corollary 2.2 it is a normal space and the Urysohn lemma can be applied to give $f_0 : \overline{V} \to [0, 1]$ with $f_0 \equiv 1$ on K and and $f_0 \equiv 0$ on $\overline{V} \setminus V$. Then of course the function f_1 defined by $f_1 \equiv f_0$ on $\overline{V}$ and $f_1 \equiv 0$ on $X \setminus \overline{V}$ is continuous (check!) because $f|\overline{V}$ is continuous and f is identically zero (the value of $f|X \setminus \overline{V}$) on the overlap set $\overline{V} \setminus V \equiv \overline{V} \cap (X \setminus V)$. Finally we let $f \equiv 2\min\{f_1, \frac{1}{2}\}$ and observe that f is then identically 1 in the set where $f_1 > \frac{1}{2}$, which is an open set containing K, and f evidently has all the remaining stated properties. □

The following corollary of Lemma 2.4 is important:

2.5 Corollary (Partition of Unity.) *If X is a locally compact Hausdorff space, $K \subset X$ is compact, and if $U_1, \ldots, U_N$ is any open cover for K, then there exist continuous $\varphi_j : X \to [0, 1]$ such that $\operatorname{spt} \varphi_j$ is a compact subset of U_j for each j, and $\sum_{j=1}^N \varphi_j \equiv 1$ in a neighborhood of K.*

Proof: By Lemma 2.3, for each $x \in K$ there is a $j \in \{1, \ldots, N\}$ and a neighborhood U_x of x such that $\overline{U}_x$ is a compact subset of this U_j. By compactness of K we have finitely many of these neighborhoods, say $U_{x_1}, \ldots, U_{x_N}$ with $K \subset \cup_{i=1}^N U_{x_i}$. Then for each $j = 1, \ldots, N$ we define V_j to be the union of all U_{x_i} such that $\overline{U}_{x_i} \subset U_j$. Then the $\overline{V}_j$ is a compact subset of U_j for each j, and the V_j cover K. So by Lemma 2.4 for each $j = 1, \ldots, N$ we can select $\psi_j : X \to [0, 1]$ with $\psi_j \equiv 1$ on $\overline{V}_j$ and $\psi_j \equiv 0$ on $X \setminus W_j$ for some open W_j with $\overline{W}_j$ a compact subset of U_j and $W_j \supset \overline{V}_j$. We can also use Lemma 2.4 to select $\varphi_0 : X \to [0, 1]$ with $\varphi_0 \equiv 0$ in a neighborhood of K and $\varphi_0 \equiv 1$ outside a compact subset of $\cup_{j=1}^N V_j$. Then by construction $\sum_{i=0}^N \psi_i > 0$ everywhere on X, so we can define continuous functions φ_j by

$$\varphi_j = \frac{\psi_j}{\sum_{i=0}^N \psi_i}, \quad j = 1, \ldots, N.$$

Evidently these functions have the required properties. □

We now give the definition of Radon measure. Radon measures are typically used only in locally compact Hausdorff space, but the definition and the first two lemmas following it are valid in arbitrary Hausdorff space:

2.6 Definition: Given a Hausdorff space X, a "Radon measure" on X is an outer measure μ on X having the 3 properties:

$$\mu \text{ is Borel regular and } \mu(K) < \infty \ \forall \text{ compact } K \subset X \tag{R1}$$

$$\mu(A) = \inf_{U \text{ open}, U \supset A} \mu(U) \text{ for each subset } A \subset X \tag{R2}$$

$$\mu(U) = \sup_{K \text{ compact}, K \subset U} \mu(K) \text{ for each open } U \subset X. \tag{R3}$$

Such measures automatically have a property like (R3) with an arbitrary μ-measurable subset of finite measure:

2.7 Lemma. *Let X be a Hausdorff space and μ a Radon measure on X. Then μ automatically has the property*

$$\mu(A) = \sup_{K \subset A,\, K \text{ compact}} \mu(K)$$

for every μ-measurable set $A \subset X$ with $\mu(A) < \infty$.

Proof: Let $\varepsilon > 0$. By definition of Radon measure we can choose an open U containing A with $\mu(U \setminus A) < \varepsilon$, and then a compact $K \subset U$ with $\mu(U \setminus K) < \varepsilon$ and finally an open W containing $U \setminus A$ with $\mu(W \setminus (U \setminus A)) < \varepsilon$ (so that $\mu(W) \le \varepsilon + \mu(U \setminus A) < 2\varepsilon$). Then we have that $K \setminus W$ is a compact subset of $U \setminus W$, which is a subset of A, and also

$$\mu(A \setminus (K \setminus W)) \le \mu(U \setminus (K \setminus W)) \le \mu(U \setminus K) + \mu(W) \le 3\varepsilon,$$

which completes the proof. □

The following lemma asserts that the defining property (R1) of Radon measures follows automatically from the remaining two properties ((R2) and (R3)) in case μ is finite and additive on finite disjoint unions of compact sets.

2.8 Lemma. *Let X be a Hausdorff space and assume that μ is an outer measure on X satisfying the properties* (R2), (R3) *above, and in addition assume that*

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) < \infty, \; K_1, K_2 \text{ compact}, \; K_1 \cap K_2 = \emptyset.$$

Then μ is Borel regular, hence (R1) *holds, hence μ is a Radon measure.*

Proof: Note that (R2) implies that for every set $A \subset X$ we can find open sets U_j such that $A \subset \cap_j U_j$ and $\mu(A) = \mu(\cap_j U_j)$. So to complete the proof of (R1) we just have to check that all Borel sets are μ-measurable; since the μ-measurable sets form a σ-algebra and the Borel sets form the smallest σ-algebra which contains all the open sets, we thus need only to check that all open sets are μ-measurable.

Let $\varepsilon > 0$ be arbitrary, Y an arbitrary subset of X with $\mu(Y) < \infty$ and let U be an arbitrary open subset of X. By (R2) we can pick an open set $V \supset Y$ with $\mu(V) < \mu(Y) + \varepsilon$ and by (R3) we can pick a compact set $K_1 \subset V \cap U$ with $\mu(V \cap U) \le \mu(K_1) + \varepsilon$, and then a compact set $K_2 \subset V \setminus K_1$ with

$\mu(V \setminus K_1) \le \mu(K_2) + \varepsilon$. Then

$$\begin{aligned}
\mu(V \setminus U) + \mu(V \cap U) &\le \mu(V \setminus K_1) + \mu(K_1) + \varepsilon \\
&\le \mu(K_2) + \mu(K_1) + 2\varepsilon \\
&= \mu(K_2 \cup K_1) + 2\varepsilon \qquad \text{(by (i))} \\
&\le \mu((V \setminus K_1) \cup K_1) + 2\varepsilon \\
&= \mu(V) + 2\varepsilon \le \mu(Y) + 3\varepsilon,
\end{aligned}$$

hence $\mu(Y \setminus U) + \mu(Y \cap U) \le \mu(V \setminus U) + \mu(V \cap U) \le \mu(Y) + 3\varepsilon$ which by arbitrariness of ε gives $\mu(Y \setminus U) + \mu(Y \cap U) \le \mu(Y)$, which establishes the μ-measurability of U. Thus all open sets are μ-measurable, and hence all Borel sets are μ-measurable, and so (R1) is established. □

The following lemma guarantees the convenient fact that, in a locally compact space such that all open subsets are σ-compact, all locally finite Borel regular outer measures are in fact Radon measures.

2.9 Lemma. *Let X be a locally compact Hausdorff space and suppose that each open set is the countable union of compact subsets. Then any Borel regular outer measure on X which is finite on each compact set is automatically a Radon measure.*

Proof: First observe that in a Hausdorff space X the statement "each open set is the countable union of compact subsets" is equivalent to the statement "X is σ-compact (i.e. the countable union of compact sets) and every closed set is the countable intersection of open sets" as one readily checks by using De Morgan's laws and the fact that a set is open if and only if its complement is closed. Thus we have at our disposal the facts that X is σ-compact and every closed set is a countable intersection of open sets. The latter fact enables us to apply Theorem 1.1 (1), and we can therefore assert that

(1) $\mu(A) = \inf_{U \text{ open},\, A \subset U} \mu(U)$ whenever $A \subset X$ has the property that $\exists$ open V_j with $A \subset \cup_j V_j$ and $\mu(V_j) < \infty \; \forall\, j$.

Also, by applying Theorem 1.1 (2) to the finite Borel regular measure $\mu \llcorner A$,

(2) $\mu(A) = \sup_{C \text{ closed},\, C \subset A} \mu(C)$, provided A is μ-measurable and $\mu(A) < \infty$.

Now observe that by the first part of the conclusion in Lemma 2.4 there is an open set $V \supset K$ such that $\overline{V}$ (the closure of V) is compact. So since we are given $X = \cup_{j=1}^{\infty} K_j$, where each K_j is compact, we can apply this with K_j in place of K, and we deduce that there are open sets V_j in X such that $\cup_j V_j = X$

and $\mu(V_j) < \infty$ for each j, and so in this case (when X is σ-compact) the identity in (1) holds for every subset $A \subset X$; that is

$$\mu(A) = \inf_{U \text{ open}, A \subset U} \mu(U) \text{ for every } A \subset X,$$

which is the property (R2). Next we note that if $A \subset X$ is μ-measurable, then we can write $A = \cup_j A_j$, where $A_j = A \cap K_j$ (because $X = \cup_j K_j$) and $\mu(A_j) \le \mu(K_j) < \infty$ for each j, so (2) actually holds for every μ-measurable A in case X is σ-compact (i.e. in case $X = \cup_{j=1}^{\infty} K_j$ with K_j compact), and for any closed set C we can write $C = \cup_j C_j$ where C_j is the increasing sequence of compact sets given by $C_j = C \cap (\cup_{i=1}^{j} K_i)$ and so $\mu(C) = \lim_j \mu(C_j)$ and hence $\mu(C) = \sup_{K \subset C,\, K \text{ compact}} \mu(K)$. Thus in the σ-compact case (2) actually tells us that $\mu(A) = \sup_{K \subset A,\, K \text{ compact}} \mu(K)$ for any μ-measurable set A. This in particular holds for $A =$ an open set, which is the remaining property (R3) we needed. □

The following result is one of the main theorems related to Radon measures, asserting that for a Radon measure μ on a locally compact Hausdorff space, the continuous functions with compact support are dense in $L^p(\mu)$, $1 \le p < \infty$.

Here and subsequently we use the notation

$$C_c(X) = \{f : X \to \mathbb{R} : f \text{ is continuous with } \operatorname{spt} f \text{ compact}\},$$

where $\operatorname{spt} f =$ support of $f =$ closure of $\{x \in X : f(x) \neq 0\}$.

2.10 Theorem. *Let X be a locally compact Hausdorff space, μ a Radon measure on X and $1 \le p < \infty$. Then $C_c(X)$ is dense in $L^p(\mu)$; that is, for each $\varepsilon > 0$ and each $f \in L^p$ there is a $g \in C_c(X)$ such that $\|g - f\|_p < \varepsilon$.*

In view of Lemma 2.9 and the fact that to every Borel measure μ on a topological space X (i.e. every map $\mu : \{\text{all Borel sets of } X\} \to [0, \infty]$ with $\mu(\emptyset) = 0$ and $\mu(\cup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \mu(B_j)$ for every pairwise disjoint collection of Borel sets $B_1, B_2, \ldots$), there is a Borel regular outer measure $\overline{\mu}$ on X defined by $\overline{\mu}(A) = \inf_{B \text{ Borel}, B \supset A} \mu(B)$, we see that Theorem 2.10 directly implies the following important corollary:

2.11 Corollary. *If X is a locally compact Hausdorff space such that every open set in X is the countable union of compact sets, and if μ is any Borel measure on X which is finite on each compact set, then the space $C_c(X)$ is dense in $L^1(\mu)$ and μ is the restriction to the Borel sets of a Radon measure $\overline{\mu}$.*

Proof of Theorem 2.10: Let $f : X \to \mathbb{R}$ be μ-measurable with $\|f\|_p < \infty$ and let $\varepsilon > 0$. Observe that the simple functions are dense in $L^p(\mu)$ (which one can check using the dominated convergence theorem and the fact that both f_+ and f_- can be expressed as the pointwise limits of increasing sequences of non-negative simple functions), so we can pick a simple function $\varphi = \sum_{j=1}^{N} a_j \chi_{A_j}$, where the a_j are distinct non-zero reals and A_j are pairwise disjoint μ-measurable subsets of X, such that $\|f - \varphi\|_p < \varepsilon$. Since $\|\varphi\|_p \le \|\varphi - f\|_p + \|f\|_p < \infty$ we must then have $\mu(A_j) < \infty$ for each j. Pick $M > \max\{|a_1|, \ldots, |a_N|\}$ and use Lemma 2.7 to select compact $K_j \subset A_j$ with $\mu(A_j \setminus K_j) < \varepsilon^p/(2^{p+1} M^p N)$. Also, using the definition of Radon measure, we can find open $U_j \supset K_j$ with $\mu(U_j \setminus K_j) < \varepsilon^p/(2^{p+1} M^p N)$ and by Lemma 2.7 we can assume without loss of generality that these open sets $U_1, \ldots, U_N$ are pairwise disjoint (otherwise replace U_j by $U_j \cap U_j^0$, where $U_1^0, \ldots, U_N^0$ are pairwise disjoint open sets with $K_j \subset U_j^0$). By Lemma 2.4 we have $g_j \in C_c(X)$ with $g_j \equiv a_j$ on K_j, $\{x : g_j(x) \neq 0\}$ contained in a compact subset of U_j, and $\sup |g_j| \le |a_j|$, and hence by the pairwise disjointness of the U_j we have that $g \equiv \sum_{j=1}^{N} g_j$ agrees with φ on each K_j and $\sup |g| = \sup |\varphi| < M$. Then $\varphi - g$ vanishes off the set $\cup_j ((U_j \setminus K_j) \cup (A_j \setminus K_j))$ and we have $\int_X |\varphi - g|^p \, d\mu \le \sum_j \int_{(U_j \setminus K_j) \cup (A_j - K_j)} |\varphi - g|^p \, d\mu \le (2M)^p \sum_j (\mu(A_j \setminus K_j) + \mu(U_j \setminus K_j)) \le \varepsilon^p$, and hence $\|f - g\|_p \le \|f - \varphi\|_p + \|\varphi - g\|_p \le 2\varepsilon$, as required.

We now state the Riesz representation theorem for non-negative functionals on the space $\mathcal{K}_+$, where, here and subsequently, $\mathcal{K}_+$ denotes the set of non-negative $C_c(X, \mathbb{R})$ functions, i.e. the set of continuous functions $f : X \to [0, \infty)$ with compact support.

2.12 Theorem (Riesz for non-negative functionals.) *Suppose X is a locally compact Hausdorff space, $\lambda : \mathcal{K}_+ \to [0, \infty)$ with $\lambda(cf) = c\lambda(f)$, $\lambda(f + g) = \lambda(f) + \lambda(g)$ whenever $c \ge 0$ and $f, g \in \mathcal{K}_+$, where $\mathcal{K}_+$ is the set of all non-negative continuous functions f on X with compact support. Then there is a Radon measure μ on X such that $\lambda(f) = \int_X f \, d\mu$ for all $f \in \mathcal{K}_+$.*

Before we begin the proof of 2.12 we the following preliminary observation:

2.13 Remark: Observe that if $f, g \in \mathcal{K}_+$ with $f \le g$ then $g - f \in \mathcal{K}_+$ and hence $\lambda(g) = \lambda(f + (g - f)) = \lambda(f) + \lambda(g - f) \ge \lambda(f)$, so

$$(*) \qquad f, g \in \mathcal{K}_+ \text{ with } g \equiv 1 \text{ on } \operatorname{spt} f \Rightarrow \lambda(f) \le (\sup f)\, \lambda(g), \quad f \in \mathcal{K}_+,\ \operatorname{spt} f \subset K.$$

because $fg \equiv f$ and $f \le (\sup f) g$.

Proof of Theorem 2.12: For $U \subset X$ open, we define

$$(1) \qquad \mu(U) = \sup_{f \in \mathcal{K}_+, f \le 1, \operatorname{spt} f \subset U} \lambda(f),$$

and for arbitrary $A \subset X$ we define

(2) $$\mu(A) = \inf_{U \supset A, U \text{ open}} \mu(U).$$

Notice that these definitions are consistent when A is itself open, and of course the definitions (1),(2) guarantee $\mu(\emptyset) = 0$ and that μ is monotone—i.e.

(3) $$A \subset B \Rightarrow \mu(A) \le \mu(B).$$

Also if $f \in \mathcal{K}_+$ with $f \le 1$ and V is open with $V \supset \operatorname{spt} f$ then by (1) $\lambda(f) \le \mu(V)$, and hence, taking inf over such V and using (2), we see

(4) $$f \in \mathcal{K}_+ \text{ with } f \le 1 \Rightarrow \lambda(f) \le \mu(\operatorname{spt} f),$$

and then for any open U we can use (1) and (4) to conclude

(5) $$\mu(U) = \sup_{f \in \mathcal{K}_+, f \le 1, \operatorname{spt} f \subset U} \mu(\operatorname{spt} f),$$

Notice next that if K is compact then, by Lemma 2.4, if $W \supset K$ is open there is $g \in \mathcal{K}_+$ with $g \equiv 1$ in a neighborhood V of K and with $g \le 1$ and $\operatorname{spt} g \subset W$. Then by (3),(1) and ($*$) we have, for any such g,

(6) $$\mu(K) \le \mu(V) = \sup_{f \in \mathcal{K}_+, f \le 1, \operatorname{spt} f \subset V} \lambda(f) \le \lambda(g) \le \mu(W).$$

To prove that μ is an outer measure it still remains to check countable subaddivity. To see this, first let $U_1, U_2, \ldots$ be open and $U = \cup_j U_j$, then for any $f \in \mathcal{K}_+$ with $\sup f \le 1$ and $\operatorname{spt} f \subset U$ we have, by compactness of $\operatorname{spt} f$, that $\operatorname{spt} f \subset \cup_{j=1}^N U_j$ for some integer N, and by using a partition of unity $\varphi_1, \ldots, \varphi_N$ for $\operatorname{spt} f$ subordinate to $U_1, \ldots, U_N$ (see Corollary 2.5), we have $\lambda(f) = \sum_{j=1}^N \lambda(\varphi_j f) \le \sum_{j=1}^N \mu(U_j)$. Taking sup over all such f we then have $\mu(U) \le \sum_{j=1}^\infty \mu(U_j)$. It then easily follows by applying definitions (1),(2) that $\mu(\cup_j A_j) \le \sum_j \mu(A_j)$. So indeed μ is an outer measure on X.

Finally we want to show that μ is a Radon measure. For this we are going to use Lemma 2.8 above, so we have to check the hypotheses of Lemma 2.8. Hypothesis (R2) needed for Lemma 2.8 is true by definition and (R3) is true by (5). Since we also have finiteness of $\mu(K)$ for compact K by (6), it remains only to prove the additivity property

(7) K_1, K_2 disjoint compact sets in $X \Rightarrow \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2)$.

To check this, let U be any open set containing $K_1 \cup K_2$ and use Corollary 2.2 to choose disjoint open $V_j \supset K_j$ with $V_j \subset U$, $j = 1, 2$. Then by (3) and (1)

$$\begin{aligned}\mu(K_1) + \mu(K_2) \le \mu(V_1) + \mu(V_2) &= \sup_{g_j \in \mathcal{K}_+, \operatorname{spt} g_j \subset V_j, g_j \le 1, j=1,2} (\lambda(g_1) + \lambda(g_2)) \\ &= \sup_{g_j \in \mathcal{K}_+, \operatorname{spt} g_j \subset V_j, g_j \le 1, j=1,2} \lambda(g_1 + g_2)\end{aligned}$$

On the other hand $g_1 + g_2 \le 1$ on X (because $V_1 \cap V_2 = \emptyset$) and so

$$\sup_{g_j \in \mathcal{K}_+, \operatorname{spt} g_j \subset V_j, g_j \le 1, j=1,2} \lambda(g_1 + g_2) \le \sup_{f \in \mathcal{K}_+, \operatorname{spt} f \subset U, f \le 1} \lambda(f) = \mu(U).$$

Hence we have proved that $\mu(K_1) + \mu(K_2) \le \mu(U)$, and taking inf over all open $U \supset K_1 \cup K_2$ we have by (2) that $\mu(K_1) + \mu(K_2) \le \mu(K_1 \cup K_2)$, and of course the reverse inequality holds by subadditivity of μ, hence the hypotheses of Lemma 2.8 are all established and μ is a Radon measure.

Next observe that by (4) we have $\lambda(h) \le \mu(\operatorname{spt} h) \sup h$, $h \in \mathcal{K}_+$, and hence

$$\lambda(h) = \lim_{n \to \infty} \lambda(\max\{h - 1/n, 0\}) \le \mu(\{x : h(x) > 0\}) \sup h, \ h \in \mathcal{K}$$

since h is the uniform limit of $\max\{h - 1/n, 0\}$ in X and $\operatorname{spt} \max\{h - 1/n, 0\} \subset \{x : h(x) > 0\}$ for each n. For $f \in \mathcal{K}_+$ (f not identically zero) and $\varepsilon > 0$, we let $M = \sup f$ can select points $0 = t_0 < t_1 < t_2 < \ldots < t_{N-1} < M < t_N$ with $t_j - t_{j-1} < \varepsilon$ for each $j = 1, \ldots, N$ and with $\mu(\{f^{-1}\{t_j\}\}) = 0$ for each $j = 1, \ldots, N$. Notice that the latter requirement is no problem because $\mu(\{f^{-1}\{t\}\}) = 0$ for all but a countable set of $t > 0$, by virtue of the fact that $\mu\{x \in X : f(x) > 0\} \le \mu(\operatorname{spt} f) < \infty$.

Now let $U_j = f^{-1}\{(t_{j-1}, t_j)\}$, $j = 1, \ldots, N$. (Notice that then the U_j are pairwise disjoint and each $U_j \subset K$, where K, compact, is the support of f.) Now by the definition (1) we can find $g_j \in \mathcal{K}_+$ such that $g_j \le 1$, $\operatorname{spt} g_j \subset U_j$, and $\lambda(g_j) \ge \mu(U_j) - \varepsilon/N$. Also for any compact $K_j \subset U_j$ we can construct a function $h_j \in \mathcal{K}_+$ with $h_j \equiv 1$ in a neighborhood of $K_j \cup \operatorname{spt} g_j$, $\operatorname{spt} h_j \subset U_j$, and $h_j \le 1$ everywhere. Then $h_j \ge g_j$, $h_j \le 1$ everywhere and $\operatorname{spt} h_j$ is a compact subset of U_j and so

(9) $$\mu(U_j) - \varepsilon/N \le \lambda(g_j) \le \lambda(h_j) \le \mu(U_j), \quad j = 1, \ldots, N.$$

Since μ is a Radon measure, we can in fact choose the compact $K_j \subset U_j$ such that $\mu(U_j \setminus K_j) < \varepsilon/N$. Then, because $\{x : (f - f\sum_{j=1}^N h_j)(x) > 0\} \subset \cup(U_j \setminus K_j)$, by (8) we have

(10) $$\lambda(f - f\textstyle\sum_{j=1}^N h_j) \le M \sum_{j=1}^N \mu(U_j \setminus K_j) \le \varepsilon M.$$

Then by using (9), (10) and the linearity of λ (together with the fact $t_{j-1} h_j \le f h_j \le t_j h_j$) for each $j = 1, \ldots, N$), we see that

$$\begin{aligned}\textstyle\sum_{j=1}^N t_{j-1} \mu(U_j) - \varepsilon M \le \lambda(f \sum_j h_j) \le \lambda(f) &\le \lambda(f \textstyle\sum_j h_j) + \varepsilon M \\ &\le \textstyle\sum_{j=1}^N t_j \mu(U_j) + \varepsilon M.\end{aligned}$$

Since trivially

$$\textstyle\sum_{j=1}^N t_{j-1} \mu(U_j) \le \int_X f \, d\mu \le \sum_{j=1}^N t_j \mu(U_j),$$

we then have

$$\begin{aligned}-\varepsilon(\mu(K)+M) &\le -\textstyle\sum_{j=1}^N (t_j-t_{j-1})\mu(U_j)-\varepsilon M\\ &\le \textstyle\int_X f\,d\mu-\lambda(f)\\ &\le \textstyle\sum_{j=1}^N (t_j-t_{j-1})\mu(U_j)+\varepsilon M\\ &\le \varepsilon(\mu(K)+M),\end{aligned}$$

where $K=\operatorname{spt} f$. This completes the proof of 2.12. □

We can now state the Riesz Representation Theorem. In the statement, $C_c(X,\mathbb{R}^n)$ will denote the set of vector functions $f: X\to\mathbb{R}^n$ which are continuous and which have compact support. (That is $f\equiv 0$ outside a compact subset of X.)

2.14 Theorem (Riesz Representation Theorem.) *Suppose X is a locally compact Hausdorff space, and $L: C_c(X,\mathbb{R}^n)\to\mathbb{R}$ is linear with*

$$\sup_{f\in C_c(X,\mathbb{R}^n),|f|\le 1,\operatorname{spt} f\subset K} L(f)<\infty \text{ whenever } K\subset X \text{ is compact.}$$

Then there is a Radon measure μ on X such that for each compact $K\subset X$ there is a vector function $\nu: X\to\mathbb{R}^n$ with $|\nu|=1$ everywhere and ν_j μ-measurable, $j=1,\ldots,n$, and with

$$L(f)=\int_X f\cdot\nu\,d\mu \text{ for any } f\in C_c(X,\mathbb{R}^n) \text{ with } \operatorname{spt} f\subset K.$$

In the cases when X is σ-compact (i.e. $\exists$ compact $K_1,K_2,\ldots$ with $X=\cup_j K_j$) or L is bounded (i.e. $\sup_{f\in C_c(X,\mathbb{R}^n),|f|\le 1}|L(f)|<\infty$), ν can be chosen independent of K.

Proof: We first define

$$\lambda(h)=\sup_{f\in C_c(X,\mathbb{R}^n),\,|f|\le h} L(f)$$

for any $h\in\mathcal{K}_+$. We claim that λ has the linearity properties of Lemma 2.12. Indeed it is clear that $\lambda(ch)=c\lambda(h)$ for any constant $c\ge 0$ and any $h\in\mathcal{K}_+$. Now let $g,h\in\mathcal{K}_+$, and notice that if $f_1,\ f_2\in C_c(X,\mathbb{R}^n)$ with $|f_1|\le g$ and $|f_2|\le h$, then $|f_1+f_2|\le g+h$ and hence $\lambda(g+h)\ge L(f_1)+L(f_2)$. Taking sup over all such $f_1,\ f_2$ we then have $\lambda(g+h)\ge\lambda(g)+\lambda(h)$. To prove the reverse inequality we let $f\in C_c(X,\mathbb{R}^n)$ with $|f|\le g+h$, and define

$$f_1=\begin{cases}\frac{g}{g+h}f & \text{if } g+h>0\\ 0 & \text{if } g+h=0,\end{cases}\qquad f_2=\begin{cases}\frac{h}{g+h}f & \text{if } g+h>0\\ 0 & \text{if } g+h=0.\end{cases}$$

Then $f_1+f_2=f$, $|f_1|\le g$, $|f_2|\le h$ and it is readily checked that $f_1,\ f_2\in C_c(X,\mathbb{R}^n)$. Then $L(f)=L(f_1)+L(f_2)\le\lambda(g)+\lambda(h)$, and hence taking

sup over all such f we have $\lambda(g+h)\le\lambda(g)+\lambda(h)$. Therefore we have $\lambda(g+h)=\lambda(g)+\lambda(h)$ as claimed. Thus λ satisfies the conditions of the Theorem 2.12, hence there is a Radon measure μ on X such that

$$\lambda(h)=\int_X h\,d\mu,\quad h\in\mathcal{K}_+,\quad j=1,\ldots,n.$$

That is, we have

$$(\ddagger)\qquad \sup_{f\in C_c(X,\mathbb{R}^n),\,|f|\le h} L(f)=\int_X h\,d\mu,\quad h\in\mathcal{K}_+.$$

Thus if $j\in\{1,\ldots,n\}$ we have in particular (since $|fe_j|=|f|\in\mathcal{K}_+$ for any $f\in C_c(X,\mathbb{R})$) that

$$|L(fe_j)|\le\int_X |f|\,d\mu\equiv\|f\|_{L^1(\mu)}\quad \forall f\in C_c(X,\mathbb{R}).$$

Thus $L_j(f)\equiv L(fe_j)$ extends to a bounded linear functional on $L^1(\mu)$. In either of the 3 cases (i) K compact is given and we use Riesz Representation Theorem for $L^1(\mu\llcorner K)$, or (ii) $\|L\|(=\mu(X))<\infty$ and we use Riesz Representation Theorem for the finite measure case, or (iii) $X=\cup_{j=1}^\infty K_j$ with K_j compact for each j and we use Riesz Representation Theorem for the σ-finite case, we know that there is a bounded μ-measurable function ν_j such that

$$L(fe_j)=\int_X f\nu_j\,d\mu,\quad f\in C_c(X,\mathbb{R}),$$

where in case (i) we impose the additional restriction $\operatorname{spt} f\subset K$. Since any $f=(f_1,\ldots,f_n)$ can be expressed as $f=\sum_{j=1}^n f_je_j$, we thus deduce

$$(*)\qquad L(f)=\int_X f\cdot\nu\,d\mu,\quad f\in C_c(X,\mathbb{R}^n),$$

where $\nu=(\nu_1,\ldots,\nu_n)$, and so by $(\ddagger)$

$$\int_X h\,d\mu=\sup_{f\in C_c(X,\mathbb{R}^n),\,|f|\le h}\int_X f\cdot\nu\,d\mu=\sup_{f\in C_c(X,R^n),|f|=1,g\in\mathcal{K}_+,g\le h}\int_X gf\cdot\nu\,d\mu$$

for every $h\in\mathcal{K}_+$, where in case (i) we assume $\operatorname{spt} h\subset K$. Now $|f\cdot\nu|\le|f||\nu|$ so we have

$$\sup_{f\in C_c(X,R^n),|f|=1,g\in\mathcal{K}_+,g\le h}\int_X gf\cdot\nu\,d\mu\le\sup_{g\in\mathcal{K}_+,g\le h}\int_X g|\nu|\,d\mu=\int_X h|\nu|\,d\mu$$

Since $C_c(X)$ is dense in $L^1(\mu)$, we can choose a sequence f_k with $|f_k|=1$ and $f_k\cdot\nu\to|\nu|$ on $\operatorname{spt} h$, so the bound on the right of the previous inequality is attained and we have proved

$$\int_X h\,d\mu=\int_X h|\nu|\,d\mu$$

and again using the density of $C_c(X)$ in $L^1(\mu)$ we have $|\nu|=1$ μ-a.e. □

We conclude with an important compactness theorem for Radon Measures.

Recall that Alaoglu's theorem (see e.g. Royden, "Real Analysis" 3rd Edition, Macmillan 1988, p.237), which is a corollary of Tychonoff's theorem, tells us that the closed unit ball in the dual space of a normed linear space must be weak* compact: that is, given any normed linear space $\mathcal{X}$ with dual space $\mathcal{X}^*$ (i.e. $\mathcal{X}^*$ is the normed space consisting of all the bounded linear functionals $F : \mathcal{X} \to \mathbb{R}$), then $\{F \in \mathcal{X}^* : \|F\| \le 1\}$ is weak* compact, meaning that for any sequence $F_j \in \mathcal{X}^*$ with $\sup_j \|F_j\| < \infty$ there is a subsequence F_{j_k} and an $F \in \mathcal{X}^*$ with $F_{j_k}(x) \to F(x)$ for each fixed $x \in \mathcal{X}$.

In particular if X is compact and $\mathcal{X} = C(X)$ (the continuous real-valued functions on X) $\{\lambda \in \mathcal{X}^* : \|\lambda\| \le 1\}$ is weak* compact. That is, given a sequence $\{\lambda_k\}$ of bounded linear functionals on $C(X)$ with $\sup_{k\ge 1} \|\lambda_k\| < \infty$, we can find a subsequence $\{\lambda_{k'}\}$ and bounded linear functional λ such that $\lim \lambda_{k'}(f) = \lambda(f)$ for each fixed $f \in C(X)$. Using the above Riesz Representation 2.12, this implies the following assertion concerning sequences of Radon measures on X, assuming X is σ-compact.

2.15 Theorem (Compactness Theorem for Radon Measures.) *Suppose $\{\mu_k\}$ is a sequence of Radon measures on the locally compact, σ-compact Hausdorff space X with the property $\sup_k \mu_k(K) < \infty$ for each compact K. Then there is a subsequence $\{\mu_{k'}\}$ which converges to a Radon measure μ on X in the sense that*

$$\lim \int_X f \, d\mu_{k'} = \int_X f \, d\mu, \quad \textit{for each } f \in C_c(X).$$

Proof: Let $K_1, K_2, \ldots$ be an increasing sequence of compact sets with $X = \cup_j K_j$ and let $F_{j,k} : C(K_j) \to \mathbb{R}$ be defined by $F_{j,k}(f) = \int_{K_j} f \, d\mu_k$, $k = 1, 2, \ldots$. By the Alaoglu theorem there is a subsequence $F_{j,k'}$ and a non-negative bounded functional $F_j : C(K_j) \to \mathbb{R}$ with $F_{j,k'}(f) \to F_j(f)$ for each $f \in C(K_j)$. By choosing the subsequences successively and taking a diagonal sequence we then get a subsequence $\mu_{k'}$ and a non-negative linear $F : C_c(X) \to \mathbb{R}$ with $\int_X f \, d\mu_{k'} \to F(f)$ for each $f \in C_c(X)$, where $F(f) = F_j(f|K_j)$ whenever $\operatorname{spt} f \subset K_j$. (Notice that this is unambiguous because if $\operatorname{spt} f \subset K_j$ and $\ell > j$ then $F_\ell(f|K_\ell) = F_j(f|K_j)$ by construction.) Then by applying Theorem 2.12 we have a Radon measure μ on X such that $F(f) = \int_X f \, d\mu$ for each $f \in C_c(X)$, and so $\int_X f \, d\mu_{k'} \to \int_X f \, d\mu$ for each $f \in C_c(X)$.