Mathematics Department Stanford University
Math 205A Sample Final Examination Solutions

Unless otherwise indicated, you can use results covered in lecture, provided they are clearly stated.

If necessary, continue solutions on backs of pages
Note: work sheets are provided for your convenience, but will not be graded

3 Hours

Q.1	
Q.2	
Q.3	
Q.4	
Q.5	
Q.6	
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Q.10	
Q.11	
Q.12	
T/50	

Name (Print Clearly): ____________________

I understand and accept the provisions of the honor code (Signed) ____________________

1. (4 points) Let X be an topological space. Define "Borel regular outer measure on X." If μ is a Borel regular outer measure on X and if $A \subset X$ has the property that $\sup_{C \text{ closed}, C \subset A} \mu(C) = \mu(A) < \infty$, prove that A is μ-measurable.

Solution: μ is a Borel regular outer measure on X if it is an outer measure on X such that all Borel sets are μ-measurable and for each $A \subset X$ there is a Borel set $B \supset A$ with $\mu(B) = \mu(A)$.

Pick a sequence C_j of closed subsets of A with $\mu(C_j) \to \mu(A)$ and pick a Borel set $B \supset A$ with $\mu(B) = \mu(A)$. Then $C_j \subset B \forall j$ and $\mu(C_j) \to \mu(B)$, so $\mu(B \setminus (\cup_j C_j)) \le \mu(B \setminus C_j) = \mu(B) - \mu(C_j) \to 0$, hence $B = (\cup_j C_j) \cup E$, where $\mu(E) = 0$ and so $A = (\cup_j C_j) \cup (E \cap A)$, so A is the union of a Borel set and a set of measure zero, hence A is μ-measurable.

2. (4 points) If $E \subset [0,1]$ and $\lambda(E) + \lambda([0,1] \setminus E) = 1$, prove that E is λ-measurable. (Here λ is Lebesgue outer measure on $\mathbb{R}$.)

Solution: By definition of Lebesgue measure we can find open U_j with $[0,1] \setminus E \subset \cap_{j=1}^{\infty} U_j$ and $1 - \lambda(E) = \lambda([0,1] \setminus E) = \lambda(\cap U_j))$. Then by De Morgan $E \supset \cup_j K_j$, where $K_j = [0,1] \setminus U_j$ is compact and $\lambda(E) \ge 1 - \lambda(\cap U_j) \ge \lambda([0,1] \setminus ([0,1] \cap (\cap_j U_j))) = \lambda(\cup_j K_j)$ and $\cup_j K_j \subset E$, so E is measurable by (i) above.

3. (5 points) Give the definition of absolutely continuous (AC) function $f : [0,1] \to \mathbb{R}$ and prove that (i) the product fg of two AC functions $f, g : [0,1] \to \mathbb{R}$ is AC, and (ii) $\int_0^1 f\, g' = fg\big|_0^1 - \int_0^1 g\, f'$.

Solution: AC on $[0,1]$ means that for each $\varepsilon > 0$ $\exists \delta > 0$ such that $\sum_{j=1}^N |f(x_j) - f(y_j)| < \varepsilon$ whenever $[x_j, y_j]$ are p.w.d. intervals with $\sum_{j=1}^N (y_j - x_j) < \delta$ (and hence by continuity of f $\sum_{j=1}^N |f(x_j) - f(y_j)| < \varepsilon$ whenever (x_j, y_j) are p.w.d. intervals with $\sum_{j=1}^N (y_j - x_j) < \delta$). Suppose f, g are AC on $[0,1]$ and $\varepsilon > 0$. Since AC trivially implies uniform continuity we have M such that $|f|, |g| \le M$ on $[0,1]$ and we can select $\delta_1 > 0$ such that $\sum_{j=1}^N |f(x_j) - f(y_j)| < \frac{\varepsilon}{2(1+M)}$ whenever (x_j, y_j) are p.w.d. with $\sum_j (y_j - x_j) < \delta_1$ and $\delta_2 > 0$ such that $\sum_{j=1}^N |g(x_j) - g(y_j)| < \frac{\varepsilon}{2(1+M)}$ whenever (x_j, y_j) are p.w.d. with $\sum_j (y_j - x_j) < \delta_2$, so for $\delta = \min\{\delta_1, \delta_2\}$ we have $\sum_{j=1}^N |f(x_j)g(x_j) - f(y_j)g(y_j)| = \sum_{j=1}^N |f(x_j)(g(x_j) - g(y_j)) + g(y_j)(f(x_j) - f(y_j))| \le \sum_{j=1}^N M|g(x_j) - g(y_j)| + \sum_{j=1}^N M|f(x_j) - f(y_j)| < \varepsilon/2 + \varepsilon/2 = \varepsilon$ whenever (x_j, y_j) are p.w.d. with $\sum_j (y_j - x_j) < \delta$.

From lecture AC functions F on $[0,1]$ satisfy $F(1) - F(0) = \int_0^1 F'(t)\, dt$; applying this to the product $F = fg$ then gives the required indentity because $(fg)' = f'g + fg'$.

4. (4 points) Prove that f AC on $[0,1] \Rightarrow f$ is BV on $[0,1]$.

Solution: Take δ corresponding to $\varepsilon = 1$ in the definition of AC, so that $\sum_{j=1}^N |f(x_j) - f(y_j)| < 1$ whenever (x_j, y_j) are p.w.d. intervals in (0,1) with $\sum (y_j - x_j) < \delta$. Pick $P_0 : 0 = z_0 < z_1 < \cdots < z_L = 1$ be a fixed partition of $[0,1]$ with $z_j - z_{j-1} < \delta$ for each $j = 1, \ldots, L$, and let $P : 0 = w_0 < w_1 < \cdots < w_M = 1$ be an arbitrary partition of $[0,1]$. Then $P_0 \cup P$ is a partition of $[0,1]$ and $t_P(f) \le t_{P \cup P_0}(f) = \sum_{j=1}^L t_{(P_0 \cup P) \cap [z_{j-1}, z_j]}(f|[z_{j-1}, z_j]) \le L \cdot 1 = L$, so we have shown $T(f) \le L < \infty$.

5. (5 points) If μ is an outer measure on a space X, define μ-measurability (in the sense of Caratheodory) of a set $A \subset X$, and give the proof that A_1, A_2 μ-measurable $\Longrightarrow A_1 \cup A_2$ is μ-measurable.

Solution: If $Y \subset X$ is an arbitrary subset of finite μ-measure, then

$$\begin{aligned}\mu(Y \setminus (A_1 \cup A_2)) + \mu(Y \cap (A_1 \cup A_2)) &= \mu((Y \setminus A_1) \setminus A_2)) + \mu(Y \cap (A_1 \cup (A_2 \setminus A_1))) \\ &= \mu(Y \setminus A_1) - \mu((Y \setminus A_1) \cap A_2) + \mu(Y \cap (A_1 \cup (A_2 \setminus A_1))) \\ &\leq \mu(Y \setminus A_1) - \mu((Y \setminus A_1) \cap A_2) + \mu(Y \cap A_1) + \mu(Y \cap A_2 \setminus A_1) \\ &= \mu(Y \setminus A_1) + \mu(Y \cap A_1) = \mu(Y)\end{aligned}$$

where in the second line we used the measurability of A_2 and in the last line we used the measurability of A_1.

6. (3 points) Using the dominated convergence theorem or otherwise, prove that

$$\lim_{n\to\infty} \int_0^1 e^{1/x}(1+n^2x)^{-1} \sin(ne^{-1/x})\, dx = 0.$$

Solution: $|\sin y| \leq y$ for all $y \geq 0$ so $|e^{1/x}(1+n^2x)^{-1}\sin(ne^{-1/x})| \leq e^{1/x}ne^{-1/x}(1+n^2x)^{-1} = \frac{n}{1+n^2x} = \frac{1}{\sqrt{x}}\frac{n\sqrt{x}}{1+n^2x} \leq \frac{1}{\sqrt{x}}$, which is an integrable function on $(0,1)$. Also the sequence trivially converges pointwise to zero on $(0,1)$ so the dominated convergence theorem is applicable.

7. (5 points) State the 5-times covering lemma for collections $\mathcal{B}$ of closed balls contained in a bounded subset of $\mathbb{R}^n$. Using the 5-times covering lemma or otherwise, prove that if μ is a Borel measure on $\mathbb{R}^n$ such that $\liminf_{\rho\downarrow 0} \rho^{-n}\mu(B_\rho(x)) < \infty$ for μ-a.e. $x \in \mathbb{R}^n$, then μ is absolutely continuous with respect to Lebesgue measure (i.e. E Borel, $\lambda(E) = 0 \Longrightarrow \mu(E) = 0$).

Solution: If $\cup_{B\in\mathcal{B}} B$ is bounded, then there is a p.w.d. countable (or finite) subcollection $B_{\rho_j}(x_j), j = 1, 2, \ldots$ (or $j = 1, 2, \ldots, N$) with $\cup_{B\in\mathcal{B}} B \subset \cup_j B_{5\rho_j}(x_j)$.

We are given that there is a Borel set N with $\mu(N) = 0$ and $\liminf_{\rho\downarrow 0} \rho^{-n}\mu(B_\rho(x)) < \infty$ for every $x \in \mathbb{R}^n \backslash N$, and observe that then $\mathbb{R}^n \backslash N = \cup A_k$, where $A_k = \{x \in \mathbb{R}^n \backslash N : \liminf_{\rho\downarrow 0} \rho^{-n}\mu(B_\rho(x)) < k\}$. Take any bounded Borel set $E \subset \mathbb{R}^n \setminus N$ with $\lambda(E) = 0$, let $E_k = E \cap A_k$, select a bounded open set $U \supset E_k$ with $\lambda(U) < \frac{\varepsilon}{k2^k}$, and let $\mathcal{B}$ be the collection of closed balls $B_\rho(x)$ with $x \in E_k$, $B_\rho(x) \subset U$ and $\mu(B_{5\rho}(x)) < k\omega_n(5\rho)^n$. Of course $\mathcal{B}$ covers E_k and all balls in $\mathcal{B}$ are in the bounded set U, thus the 5-times covering lemma is applicable and tells us that we can select p.w.d. balls $B_{\rho_{j,k}}(x_{j,k})$ in $\mathcal{B}$ such that $B_{5\rho_{j,k}}(x_{j,k})$ covers E_k. Furthermore $\mu(\cup_j B_{5\rho_{j,k}}(x_{j,k})) \le \sum_j \mu(B_{5\rho_{j,k}}(x_{j,k})) \le 5^n k\omega_n \sum_j \rho_{j,k}^n = 5^n k \sum_j \lambda(B_{\rho_{j,k}}(x_{j,k})) = 5^n k\lambda(\cup_j B_{\rho_{j,k}}(x_{j,k})) \le 5^n k\lambda(U) \le 5^n\varepsilon 2^{-k}$ and so $\mu(E) \le \mu(\cup_{j,k} B_{5\rho_{j,k}}(x_{j,k})) \le 5^n\varepsilon$, so $\mu(E) = 0$. Since any Borel set of λ-measure zero is the countable union of bounded sets of λ-measure zero, this proves every $E \subset \mathbb{R}^n \setminus N$ has μ-measure zero. Finally if E is an arbitrary Borel set with $\lambda(E) = 0$, then we have $E = (E \setminus N) \cup (E \cap N)$ has μ-measure zero.

8. (3 points) Suppose $A \subset [0, 1]$ is dense (thus $\overline{A} = [0, 1]$) and assume $f : A \to \mathbb{R}$ has the property that $\sum_{j=1}^N |f(x_j) - f(x_{j-1})| \le 1$ whenever $N \ge 1$ and $0 \le x_0 < x_1 < \cdots < x_N \le 1$ are points of A. Prove that there is a BV function g on $[0, 1]$ which agrees with f at each point of A.

Hint: Define $g(x) = f(x)$ if $x \in A$ and $g(x) = \limsup_{y\to x,\, y\in A} f(y)$ if $x \notin A$.

Solution: Let $0 = x_0 < x_1 < \cdots < x_N = 1$ be an arbitrary partition of $[0, 1]$, and for each $i = 0, \ldots, N$ pick a sequence x_{ij} with $x_{ij} = x_i \,\forall j$ in case $x_i \in A$ and $x_{ij} \in A$ with $\lim f(x_{ij}) = \limsup_{y\to x_i, y\in A} f(y)$ if $x_i \notin A$. Observe that then $\lim_{j\to\infty} f(x_{ij}) = g(x_i)$ for each $i = 1, \ldots, N$, and for all sufficiently large j we have $0 \le x_{1j} < x_{2j} < \cdots < x_{nj} \le 1$. Since $x_{ij} \in A$ we then have $\sum_{j=1}^N |f(x_{ij}) - f(x_{i-1\,j})| \le 1$ and taking limits with respect to j we thus have $\sum_{j=1}^N |g(x_i) - g(x_{i-1})| \le 1$, so g is in BV.

9. (5 points) Let $f : \mathbb{R}^n \to \mathbb{R}$ be Lebesgue measurable. Prove that there is a Borel measurable function $g : \mathbb{R}^n \to \mathbb{R}$ with $f(x) = g(x)$ for Lebesgue a.e. $x \in \mathbb{R}^n$.

Hint: First consider the case when f is a non-negative simple function.

Solution: If $f = \sum_{j=1}^N a_j \chi_{A_j}$ where $a_j \ge 0$ and $A_1, \ldots, A_N$ are p.w.d. Lebesgue measurable sets, then from lecture we can select a Borel set $B_j \subset A_j$ with $\lambda(A_j \setminus B_j) = 0$, and hence $g = \sum_{j=1}^N a_j \chi_{B_j}$ is Borel measurable, $\le f$ everywhere, and agrees with f except at $\cup_{j=1}^N (A_j \setminus B_j)$, which is a set of measure zero. Thus the result is proved in case f is a non-negative simple function. If f is a non-negative Lebesgue measurable function, then we can find an increasing sequence of non-negative Lebesgue measurable simple functions s_j with $s_j \to f$ at each point of $\mathbb{R}^n$. According to the first case discussed above we can then find a non-negative Borel measurable simple function t_j with $t_j \le s_j$ and $t_j = s_j$ except on a set E_j of Lebesgue measure zero. Then $t_j \le f$ everywhere and $t_j \to f$ at each point of $\mathbb{R}^n \setminus (\cup_j E_j)$. Thus $\limsup_{j\to\infty} t_j$ is a Borel measurable real-valued function which agrees a.e. with f. Finally if f has arbitrary sign, then we can write $f = f_+ - f_-$ and apply the result just established to each of f_+, f_-.

10. (5 points) Let $\mathcal{N} = \{1, 2, \ldots\}$, let $\mathcal{A}$ be the collection of all subsets of $\mathcal{N}$ and let μ be the counting measure on $\mathcal{N}$ (i.e. $\mu(A) =$ number of elements in the set A, taken to be 0 if $A = \emptyset$ and ∞ if A is an infinite subset.) If $f : \mathcal{N} \to \mathbb{R}$ describe (in terms of series terminology, taking $a_n = f(n)$) (i) Assuming $f \ge 0$, find the value of $\int_{\mathcal{N}} f\,d\mu$ by directly applying the definition of the integral, (ii) Find, in series terminology, what it means for f to be μ integrable, (iii) Using only series terminology state the monotone convergence theorem and the dominated convergence theorem in this setting, and give the proof of each using a direct argument without reference to measure theory.

Solution Of course all $f : \mathcal{N} \to \mathbb{R}$ are $\mathcal{A}$-measurable, and for a non-negative function $f : \mathcal{N} \to [0, \infty]$ the definition of the integral

$$\begin{aligned}\int_{\mathcal{N}} f\,d\mu &= \sup_{\text{simple functions } \varphi:\mathcal{N}\to[0,\infty) \text{ with } \varphi\le f} \int_{\mathcal{N}} \varphi \\ &= \sup_{\text{non-negative } c_1,\ldots,c_N \text{ with } c_n\le f(n)\forall n=1,\ldots,N} \sum_{n=1}^N c_n = \sup_N \sum_{n=1}^N f(n) = \sum_{n=1}^\infty f(n).\end{aligned}$$

(ii) f is μ-integrable means $\int_{\mathcal{N}} |f|\,d\mu < \infty$, which is exactly the condition $\sum_{n=1}^\infty |f(n)| < \infty$ (i.e. that $\sum_{n=1}^\infty f(n)$ is AC).

(iii) The monotone convergence theorem says that if f_j is an increasing sequence of functions $\mathcal{N} \to [0, \infty]$ then $\int_{\mathcal{N}} \lim f_j\,d\mu = \lim_j \int_{\mathcal{N}} f_j\,d\mu$, which in series terminology says that for non-negative a_n, a_{jn} with $a_{jn} \to a_n$ it is true that $\lim_{j\to\infty} \sum_n a_{jn} = \sum_n a_n$. To prove it note that for each N we have $\sum_{n=1}^N a_n = \lim_{j\to\infty} \sum_{n=1}^N a_{jn} \le \sum_{n=1}^N a_n$. Hence letting $N \to \infty$ we have $\sum_{n=1}^\infty a_n \le \liminf_{j\to\infty} \sum_{n=1}^\infty a_{jn} \le \limsup_{j\to\infty} \sum_{n=1}^\infty a_{jn} \le \sum_{n=1}^\infty a_n$.

The dominated convergence theorem says that if $f, f_j, g : \mathcal{N} \to \mathbb{R}$ and if $|f_j(n)| \le g(n)$ for each n with $\sum_n g_n < \infty$ and $\lim_{j\to\infty} f_j(n) = f(n)$ for each n, then $\lim_j \sum_n a_{jn} = \sum_n a_n$, where $a_{jn} = f_j(n)$ and $a_n = f(n)$. To prove this let $b_n = g(n)$ and note that $\lim_j \sum_{n=1}^N a_{jn} = \sum_{n=1}^N a_n$ for each N and on the other hand by the comparison test $\sup_j \sum_{n=N+1}^\infty (|a_n|+|a_{jn}|) \le 2\sum_{n=N+1}^\infty b_n \to 0$ as $N \to \infty$. So $\sum_n a_{jn}, \sum_n a_n$ are convergent and if $\varepsilon > 0$ there is N such that $|\sum_n a_n - \sum_n a_{jn}| \le |\sum_{n=1}^N a_n - \sum_{n=1}^N a_{jn}| + |\sum_{n=N+1}^\infty a_n - \sum_{n=N+1}^\infty a_{jn}| \le |\sum_{n=1}^N a_n - \sum_{n=1}^N a_{jn}| + \varepsilon$ for all $j \ge N$. Since $\lim_{j\to\infty}(\sum_{n=1}^N a_n - \sum_{n=1}^N a_{jn}) = 0$ we thus have $|\sum_n a_n - \sum_n a_{jn}| < 2\varepsilon$ for all j sufficiently large.

11. (4 points) Suppose $(X, \mathcal{A}, \mu)$ is any σ-finite measure space, $1 < p < \infty$, and $g : X \to \mathbb{R}$ is $\mathcal{A}$-measurable such that there is a constant $C > 0$ with$\int_X |fg|\, d\mu \le C\|f\|_p$ for each $f \in \mathcal{L}^p(\mu)$. Prove that $g \in \mathcal{L}^q(\mu)$, where q is the conjugate exponent to p (i.e. $1/p + 1/q = 1$).

Solution: Let $B_1, B_2, \ldots$ be an increasing sequence in $\mathcal{A}$ with $\mu(B_k) < \infty$ for each k and $X = \cup_k B_k$, and let $f_k = \chi_{E_k} \operatorname{sgn}(g)|g|^{q/p}$, where $E_k = \{x \in B_k : |g| < k\}$. Then $\|f_k\|_p = \int_{E_k} |g|^q \le k\mu(B_k) < \infty$ for each k, so $f_k \in \mathcal{L}^p(\mu)$ and hence $\int_{E_k} |g|^q \le C(\int_{E_k} |g|^q\, d\mu)^{1/p}$ which gives $\|\chi_{E_k} g\|_q \le C$, and, by letting $k \to \infty$ and using the Monotone convergence, theorem we have $\|g\|_q \le C < \infty$.

Alternative Solution: $T(\widetilde{f}) = \int_X gf\, d\mu$ is evidently a bounded linear functional on $L^p(\mu)$, so by the Riesz theorem there is $g_0 \in \mathcal{L}^q(\mu)$ with $T(\widetilde{f}) = \int_X f g_0\, d\mu$. That is $\int_X f(g - g_0)\, d\mu = 0$ for each $f \in \mathcal{L}^p(\mu)$. Let $B_1, B_2, \ldots$ be an increasing sequence in $\mathcal{A}$ with $\mu(B_k) < \infty$ and $\cup_k B_k = X$, and observe that $\chi_{B_k} \operatorname{sgn}(g - g_0) \in \mathcal{L}^p(\mu)$, hence we can choose $f = \chi_{B_k} \operatorname{sgn}(g - g_0)$ in the above identity giving $\int_{B_k} |g - g_0|\, d\mu = 0$ and hence $g = g_0$ μ-a.e. on B_k, hence $g = g_0$ μ-a.e. on X.

12. (3 points) Let $(X, \mathcal{A}, \mu)$, $(Y, \mathcal{B}, \nu)$ be arbitrary measure spaces and let γ be the product outer measure defined as usual by $\gamma(A) = \inf \sum_i \mu(A_i)\nu(B_i)$ with the inf over all collections $\{A_i \times B_i\}_{i=1,2,\ldots}$ with $A_i \in \mathcal{A}$, $B_i \in \mathcal{B}$, and $A \subset \cup_i A_i \times B_i$.

Prove that $\gamma(\cup_j E_j \times F_j) = \sum_j \mu(E_j)\nu(F_j)$ whenever $E_j \in \mathcal{A}$, $F_j \in \mathcal{B}$ and the sets $E_1 \times F_1, E_2 \times F_2, \ldots$ are pairwise disjoint.

Note: Your proof should not depend on Fubini's theorem—recall that the above lemma was proved as part of the preliminary discussion needed in the eventual proof of Fubini's theorem.

Solution: By definition of γ we of course have $\gamma(\cup_j E_j \times F_j) \le \sum_j \mu(E_j)\nu(F_j)$, so we just need the reverse inequality. Let $A_i \times B_i$ be any cover for $\cup_j E_j \times F_j$; then $\sum_i \chi_{A_i}(x)\chi_{B_i}(y) = \sum_i \chi_{A_i \times B_i}(x, y) \ge \chi_{\cup_i A_i \times B_i}(x, y) \ge \chi_{\cup_j E_j \times F_j}(x, y) = \sum_j \chi_{E_j \times F_j}(x, y) = \sum_j \chi_{E_j}(x)\chi_{F_j}(y)$, and hence in particular $\sum_j \chi_{E_j}(x)\chi_{F_j}(y) \le \sum_i \chi_{A_i}(x)\chi_{B_i}(y)$. Holding x fixed and integrating with respect to y we thus have $\sum_j \chi_{E_j}(x)\nu(F_j) \le \sum_i \chi_{A_i}\nu(B_i)$, and then integrating with respect to x we get $\sum_j \mu(E_j)\nu(F_j) \le \mu(A_i)\nu(B_i)$ and then taking the inf over all such collections $A_i \times B_i$ we get $\sum_j \mu(E_j)\nu(F_j) \le \gamma(\cup_i E_i \times F_i)$ as required.