Math 20: Midterm 2

Monday, 02/22/2010

- Complete the following problems. You may use any result from class you like, but if you cite a theorem be sure to verify the hypotheses are satisfied.
- This is a closed-book, closed-notes exam. No calculators or other electronic aids will be permitted.
- In order to receive full credit, *you must show all of your work and justify your answers.* Your answer should be clearly labeled.
- If you need extra room, use the back sides of each page. Staple any scratch paper to your exam.
- Here are some useful formulas:
  - $\sin(2x) = 2\sin(x)\cos(x)$
  - $\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1 = 1 - 2\sin^2(x)$
- Please sign the following:

  "On my honor, I have neither given nor received any aid on this examination. I have furthermore abided by all other aspects of the honor code with respect to this examination.

Name: Solutions

Signature: ______________________

(1) ____________ (/30 points)

(2) ____________ (/10 points)

(3) ____________ (/15 points)

(4) ____________ (/10 points)

(5) ____________ (/15 points)

(6) ____________ (/10 points)

Total. ____________ (/90 points)

(1) (30 points) For each of the problems below, integrate. If the integral is improper, interpret accordingly. If the integral is indefinite, you *must* check your answer by differentiating (this is worth 1 point). (Note: it is possible to do all of these problems, do not resort to approximate integration)

(a)

$$\int \cos(3x) + \sqrt{3x}\, dx$$

let $u = 3x \quad du = 3\,dx \rightarrow dx = \frac{1}{3} du$

$$\int \cos(3x) + \sqrt{3x}\, dx = \frac{1}{3}\int \cos(u) + u^{1/2}\, du = \frac{1}{3}\left[\sin(u) + \frac{2}{3}u^{3/2}\right] + C$$

$$= \frac{1}{3}\sin(3x) + \frac{2}{9}(3x)^{3/2} + C$$

check: $F(x) = \frac{1}{3}\sin(3x) + \frac{2}{9}(3x)^{3/2} + C$

$F'(x) = \frac{1}{3}\cos(3x)\cdot 3 + \frac{2}{9}\cdot\frac{3}{2}(3x)^{1/2}\cdot 3 + C$

$= \cos(3x) + (3x)^{1/2}$ ✓.

(b)

odd even odd even

$$\int_{-1}^{1} x^3 + 3x^2 + x + 1\, dx$$

symmetric interval

$$\int_{-1}^{+1} x^3 + 3x^2 + x + 1\, dx = 2\int_0^1 3x^2 + 1\, dx$$

$$= 2\left(x^3 + x\right)\Big|_0^1$$

$$= 2(1+1)$$

$$= 4$$

(c)

$$\int 3xe^{-2x}\,dx = 3\int x e^{-2x}\,dx$$

integrate by parts $u = x$, $dv = e^{-2x}dx$
$du = dx$, $v = -\frac{1}{2}e^{-2x}$

$$3\int x e^{-2x}dx = 3\left[-\frac{1}{2}xe^{-2x} - \int -\frac{1}{2}e^{-2x}dx\right] = 3\left[-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx\right]$$

$$= 3\left[-\frac{1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + C\right] = -\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} + C$$

check: $F(x) = -\frac{3}{2}xe^{-2x} - \frac{3}{4}e^{-2x} + C$

$$F'(x) = \cancel{-\frac{3}{2}e^{-2x}} \quad -\frac{3}{2}x(-2e^{-2x}) \quad \cancel{-\frac{3}{4}(-2e^{-2x})} + 0$$

$$= 3xe^{-2x} \checkmark$$

(d)

$$\int_0^1 \frac{1}{x^2-4}\,dx$$

$$\frac{1}{x^2-4} = \frac{1}{(x-2)(x+2)} = \frac{A}{x-2} + \frac{B}{x+2}$$

$\Longrightarrow$ $1 = A(x+2) + B(x-2)$

$1 = (A+B)x + 2A - 2B$

so $A+B=0 \Longrightarrow A=-B$

$2A-2B=1 \Longrightarrow 4A=1 \quad A=\frac{1}{4},\ B=-\frac{1}{4}$

$$\int_0^1 \frac{1}{x^2-4}dx = \int_0^1 \frac{\frac{1}{4}}{x-2} + \frac{-\frac{1}{4}}{x+2}dx = \frac{1}{4}\int_0^1 \frac{1}{x-2} - \frac{1}{x+2}dx$$

$$= \frac{1}{4}\Big[\ln|x-2| - \ln|x+2|\Big]_0^1 = \frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|\Bigg]_0^1$$

$$= \frac{1}{4}\left(\ln\left|\frac{1-2}{1+2}\right| - \ln\left|\frac{-2}{2}\right|\right) = \frac{1}{4}\ln\frac{1}{3} - 0$$

(e)

$$\int_0^2 \frac{x}{x^2+1}\,dx$$

let $u = x^2+1$ when $x=0$, $u=1$

$du = 2x\,dx$ $x=2$, $u=5$

$\frac{du}{2} = x\,dx$

$$\int_0^2 \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^5 \frac{du}{u} = \frac{1}{2}\ln|u|\Big]_1^5 = \frac{1}{2}\ln 5 - \cancel{\frac{1}{2}\ln 1}^{0}$$

$$= \frac{1}{2}\ln 5$$

(f)

$$\int \frac{x}{x+1}\,dx$$

many solutions. one way $\frac{x}{x+1} = \frac{x+1-1}{x+1} = \frac{x+1}{x+1} - \frac{1}{x+1}$

$= 1 - \frac{1}{x+1}$ (or poly. divison)

So $\int \frac{x}{x+1}\,dx = \int 1 - \frac{1}{x+1}\,dx$

$= x - \ln(x+1) + C$

set $F(x) = x - \ln(x+1) + C$

$F'(x) = 1 - \frac{1}{x+1} + 0$

$= \frac{x+1-1}{x+1}$

$= \frac{x}{x+1}$ ✓.

(2) (10 points) This question is about approximate integration for the function

$$\int_1^3 e^{-x^2}\, dx.$$

(a) (5 points) Use the trapezoid rule with $n = 6$ to approximate the value of the integral. You do not need to simplify your answer.

$\Delta x = \frac{3-1}{6} = \frac{2}{6} = \frac{1}{3}$ $x_0 = 1$, $x_1 = 1 + \frac{1}{3} = \frac{4}{3}$, $x_2 = \frac{5}{3}$ $x_3 = \frac{6}{3}$

$x_4 = \frac{7}{3}$ $x_5 = \frac{8}{3}$ $x_6 = \frac{9}{3}$

$$T_6 = \frac{1}{2}\cdot\frac{1}{3}\left[e^{-1^2} + 2e^{-(\frac{4}{3})^2} + 2e^{-(\frac{5}{3})^2} + 2e^{-(\frac{6}{3})^2} + 2e^{-(\frac{7}{3})^2} + 2e^{-(\frac{8}{3})^2} + e^{-(\frac{9}{3})^2}\right]$$

(b) (5 points) Use Simpson's rule with $n = 6$ to approximate the value of the integral. You do not need to simplify your answer.

$$S_6 = \frac{1}{3}\cdot\frac{1}{3}\Big[e^{-1^2} + 4e^{-(\frac{4}{3})^2} + 2e^{-(\frac{5}{3})^2} + 4e^{-(\frac{6}{3})^2}$$
$$+ 2e^{-(\frac{7}{3})^2} + 4e^{-(\frac{8}{3})^2} + e^{-(\frac{9}{3})^2}\Big]$$

(3) (15 points) This question is about improper integrals.

(a) (5 points) Determine whether

$$\int_1^\infty \frac{1}{e^{-3x}}\,dx$$

converges or diverges.

Notice $\int \frac{1}{e^{-3x}}\,dx = \int e^{3x}\,dx = \frac{1}{3}e^{3x} + C$

So $\lim_{t\to\infty} \frac{1}{3}e^{3t} = +\infty$.

So $\int_1^\infty \frac{1}{e^{-3x}}\,dx$ diverges

(b) (5 points) Determine whether

$$\int_1^\infty \frac{1}{e^{3x}}\,dx$$

converges or diverges.

Now $\int \frac{1}{e^{3x}}\,dx = \int e^{-3x}\,dx = -\frac{1}{3}e^{-3x}$

So $\int_1^t \frac{1}{e^{3x}}\,dx = -\frac{1}{3}e^{-3x}\Big]_1^t = \frac{1}{3}\left[e^{-3} - e^{-3t}\right]$

$\lim_{t\to\infty} \frac{1}{3}\left[e^{-3} - e^{-3t}\right] = \frac{1}{3}e^{-3}$. So $\int_1^\infty \frac{1}{e^{3x}}\,dx$ converges.

(c) (5 points) Determine whether

$$\int_0^\infty \frac{1}{e^{3x}+e^{-3x}}\,dx$$

converges or diverges.

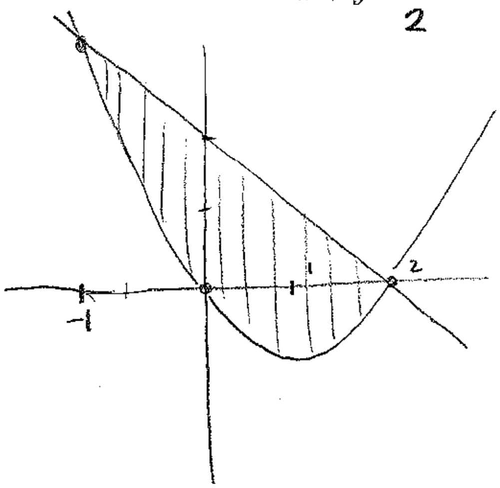
$$\int_0^\infty \frac{1}{e^{3x}+e^{-3x}}\,dx = \underbrace{\int_0^1 \frac{1}{e^{3x}+e^{-3x}}\,dx}_{\text{integrand cts on } [0,1] \text{ no problems.}} + \underbrace{\int_1^\infty \frac{1}{e^{3x}+e^{-3x}}\,dx}_{\text{does this converge?}}$$

Since $\frac{1}{e^{3x}+e^{-3x}} \le \frac{1}{e^{3x}}$ on $[1,\infty)$ and $\int_1^\infty \frac{1}{e^{3x}}\,dx$ converges by (b)

$\int_1^\infty \frac{1}{e^{3x}+e^{-3x}}\,dx$ converges by comparison. So $\int_0^\infty \frac{1}{e^{3x}+e^{-3x}}\,dx$ converges

(4) (10 points)

(a) (5 points) Sketch the region enclosed by the graphs of $y = x^2 - 2x$ and $x + y = 2$.



pts of intersection:

$$x^2 - 2x = 2 - x$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, \ x = -1$$

(b) (5 points) Compute the area of the bounded region between the graphs above.

$$A = \int_{x=-1}^{x=2} 2 - x - (x^2 - 2x)\, dx = \int_{-1}^{2} 2 + x - x^2 \, dx$$

$$= \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} = \left(4 + 2 - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right)$$

$$= 6 - \frac{8}{3} + 2 - \frac{1}{2} - \frac{1}{3}$$

$$= 3 + 2 - \frac{1}{2}$$

$$= 5 - \frac{1}{2} = \boxed{4\frac{1}{2}}$$

(5) (15 points) Answer one of the following questions. If you do both and do not mark which you want graded, the grader will grade the first solution.

A. Using techniques from class, find the area of a circle of radius 5. Show all steps.

** OR **

B. Integrate all three of the following expressions

(a) $\int \sin^2 \phi \, d\phi$

(b) $\int \frac{x^2}{\sqrt{9-x^2}} \, dx$

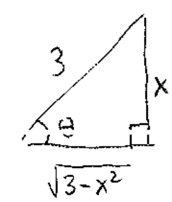
(c) $\int \frac{1}{x^2+9} \, dx$

For part (A), please see class notes.

B. (a) $\int \sin^2\phi \, d\phi = \int \frac{1-\cos(2\phi)}{2} \, d\phi$ by front page

$= \int \frac{1}{2} - \frac{\cos(2\phi)}{2} \, d\phi = \frac{1}{2}\phi - \frac{\sin(2\phi)}{4} + C$

(b) $\int \frac{x^2}{\sqrt{9-x^2}} \, dx$. let $x = 3\sin\theta$, $dx = 3\cos\theta \, d\theta$



$\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{3-x^2}}{3}$

$\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$= \int \frac{(3\sin\theta)^2}{\sqrt{9-(3\sin\theta)^2}} \cdot 3\cos\theta \, d\theta$

$= \int \frac{9\sin^2\theta}{3\cos\theta} \cdot 3\cos\theta \, d\theta$

$= 9\int \sin^2\theta \, d\theta$ by (a)

$= \frac{9}{2}\theta - \frac{9}{4}\sin(2\theta) + C$ $\sin(2\theta) = 2\sin\theta\cos\theta$

$= \frac{9}{2}\theta - \frac{9}{2}\sin(\theta)\cos(\theta) + C$ Express in terms of x

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{3-x^2}}{3}\right) + C$

$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\sqrt{3-x^2} + C$

(c) $\int \frac{1}{x^2+9} \, dx = \int \frac{1}{9\left(\frac{x^2}{9}+1\right)} dx = \frac{1}{9}\int \frac{1}{\frac{x^2}{9}+1} dx = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1} dx$

set $u = \frac{x}{3}$ $du = \frac{dx}{3}$ $dx = 3\,du$

$= \frac{1}{9}\int \frac{1}{u^2+1} \cdot 3\,du$

$= \frac{1}{3}\tan^{-1}(u) + C$

$= \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

(6) (10 points) True/False. No justification needed

(a) (2 points) True or False:

$$\int \cos^2 x \, dx = \sin^2 x + C$$

False. if $F(x) = \sin^2(x) + C$
$F'(x) = 2\sin(x)\cos(x) \neq \cos^2(x)$

(b) (2 points) True or False:

$$\int_{-10}^{10} \underbrace{x^3 + x}_{\text{odd function}} \, dx = 0$$

True.

symmetric interval

(c) (2 points) True or False: $\int_0^1 \frac{1}{x^2} \, dx$ converges

False. $\int \frac{1}{x^2} dx = \frac{-1}{x}$, and $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

(d) (2 points) True or False:

$$\int_0^2 \frac{1}{x-1} \, dx = (\ln|x-1|)_0^2 = \ln|1| - \ln|-1| = 0 - 0 = 0.$$

False.

FTC does not apply
$x=1$ is a vertical asymptote
and 1 is <u>in</u> the interval of integration.

(e) (2 points) True or False: If $g(x) = \int_0^x e^{2u} du$, then the tangent line to $y = g(x)$ at $x = 0$ is $y = x$.

True.

need $g(0) = \int_0^0 e^{2u} du = 0$

$g'(x) = e^{2x}$ by FTC

so $g'(0) = e^{2(0)} = 1$

so $y - 0 = 1 \cdot (x - 0) \implies \boxed{y = x}$