Mathematics Department Stanford University
Math 175 Homework 9

This hw will not be graded; solutions will be posted Mon June 5

1. Prove the claim made in lecture that if $f \in \mathcal{L}^2([a,b])$ then there is a sequence $\{\varphi_k\}_{k=1,2,\dots}$ of step functions on $[a,b]$ with $\|f-\varphi_k\|_2 \to 0$, where $\|\ \|_2$ denotes the $\mathcal{L}^2([a,b])$ seminorm $(\int_a^b |f-\varphi_k|^2)^{1/2}$.

Solution: We'll give a proof that only uses the monotone convergence theorem and the other theorems in the first 3 sections of the handout. Since we can write $f = f_+ - f_-$, and $f_\pm \in \mathcal{L}^2([a,b])$ if $f \in \mathcal{L}^2([a,b])$, it is enough to prove the result in case $f \geq 0$, so we assume that. Then $f_k = \min\{f, k\}$, $k = 1, 2, \dots$ is a bounded non-negative $\mathcal{L}^2([a,b])$ function, and $\|f - f_k\|_2^2 = \int_a^b (f - f_k)^2 \leq \int_a^b (f - f_k)(f + f_k) = \int_a^b (f^2 - f_k^2) \to 0$ by the monotone convergence theorem. On the other hand, for each k, by Lemma 2 of the lecture supplement there is a step function φ_k such that $\|f_k - \varphi_k\|_1 \leq 2^{-k}$, and (by replacing φ_k by the function $\min\{k, \max\{0, \varphi_k\}\}$) we can assume $0 \leq \varphi_k \leq k$ for each k, so then (since $f_k + \varphi_k \leq 2k$) $\int_a^b |f_k - \varphi_k|^2 \leq \int_a^b |f_k - \varphi_k|(f_k + \varphi_k) \leq 2k\|f_k - \varphi_k\|_1 \leq 2k2^{-k} \to 0$. Then by the triangle inequality $\|f - \varphi_k\|_2 = \|f - f_k + f_k - \varphi_k\|_2 \leq \|f - f_k\|_2 + \|f_k - \varphi_k\|_2 \to 0$.

2. Find explicitly all eigenvalues/eigenfunctions, and show zero is not an eigenvalue, of the Sturm-Liouville problem (SL) of lecture in the cases (i) $[a,b] \equiv [0,\pi]$, $p \equiv 1, \rho \equiv 1, q \equiv 0, \alpha = \gamma = 1, \beta = \delta = 0$ (so the boundary conditions are $f(0) = f(\pi) = 0$ and the equation is $-f'' = \lambda f$), and (ii) $[a,b] \equiv [0,\pi]$, $p \equiv 1, \rho \equiv 1, q \equiv -1, \alpha = \gamma = 0, \beta = \delta = 1$ (so the boundary conditions are $f'(0) = f'(\pi) = 0$ and the equation is $-f'' + f = \lambda f$).

Solution (i): As discussed in lecture, for $\lambda = 0$ any $C^2([a,b])$ solution of the equation can be written in the form $u = C_1 + C_2 x$ and there are no corresponding non-trivial solutions which satisfy the boundary conditions, so $\lambda = 0$ is not an eigenvalue. If $\lambda < 0$ we can write $\lambda = -\omega^2$ with $\omega > 0$ and in this case the general solution can be written $u(x) = C_1 \cosh \omega x + C_2 \sinh \omega x$; the boundary condition $u(0) = 0$ implies $C_1 = 0$ and then the boundary condition $u(\pi) = 0$ cannot be satisfied unless $C_2 = 0$ also. Thus again there are no non-trivial solutions. Finally if $\lambda > 0$ we can write $\lambda = \omega^2$ with $\omega > 0$ and in this case the general solution can be written $u(x) = C_1 \cos \omega x + C_2 \sin \omega x$; the boundary condition $u(0) = 0$ implies that $C_1 = 0$ and then the boundary condition $u(\pi) = 0$ is satisfied with $C_2 \neq 0$ if and only if $\omega = n$, $n = 1, 2, \dots$, so the eigenvalues are $\lambda_n = n^2$, $n = 1, 2, \dots$ and the corresponding eigenfunctions (up to non-zero scalar factors) are $\sin nx$. If we use the inner product $(f, g) = \frac{2}{\pi}\int_0^\pi f\bar{g}dx$ then $\sin nx$ has norm 1, and we know from the general Sturm-Liouville theory established in lecture that then $\{\sin nx\}_{n=1,2,\dots}$ is a complete orthonormal sequence for $L^2([0,\pi])$.

Solution (ii): Similar to (i). We will not give the details here. The conclusion is that the eigenvalues are again $\lambda_n = n^2$, $n = 1, 2, \dots$ with corresponding eigenfunctions $\frac{1}{\sqrt{2}}, \cos x, \cos 2x, \dots$, again giving a complete orthonormal system by the same general Sturm-Liouville theory.

3. Using the result of Q.2 give a proof of the completeness of the orthonormal sequence $\{e^{inx}\}_{n=0,\pm 1,\pm 2}$ in $L^2([-\pi,\pi])$ which uses the general theorem on completeness of the non-zero eigenfunctions of K in $(\ker K)^\perp$, for any compact Hermitian operator K on a Hilbert space H.

Note: Of course your proof should be independent of the Fejer kernel proof given earlier in the course

Hint: Observe that any function $f(x)$ on $[-\pi,\pi]$ can be written as the sum of an even and an odd function by writing $f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x))$.

Solution: Since $\sin nx$ is an odd function for each $n = 1, 2, \ldots$, we know from Q.2 that for any odd function $g \in \mathcal{L}^2([-\pi, \pi])$ we can write $\lim_{N\to\infty} \|g - \sum_{n=1}^{N} b_n \sin nx\|_2 = 0$, and likewise any even function $h \in \mathcal{L}^2([-\pi, \pi])$ we can write $\lim_{N\to\infty} \|g - \frac{a_0}{\sqrt{2}} - \sum_{n=1}^{N} a_n \cos nx\|_2 = 0$. So (since $\cos nx = \frac{e^{inx}+e^{-inx}}{2}$ and $\sin nx = \frac{e^{inx}-e^{-inx}}{2i}$) and since we can write $f = g + h$ with g even and h odd we have shown that $\text{span}\{e^{inx} : n = 0, \pm 1, \pm 2, \ldots\}$ are dense in L^2, and hence (by a general theorem of lecture) the orthonormal sequence $e^{inx}, n = 0, \pm 1, \pm 2, \ldots$ is complete.

4. For the general Sturm-Liouville problem (SL) of lecture, assuming 0 is not an eigenvalue, prove that all eigenvalues have multiplicity 1; that is, if λ is any eigenvalue of (SL) then span of the set of all eigenfunctions $u \in C^2([a, b])$ with eigenvalue λ is a 1-dimensional subspace of $C^2([a, b])$.

Solution: From Q.3 of hw8 we know that if u, v are both eigenfunctions corresponding to one of the eigenvalues λ_n and if u is not a constant multiple of v then we must have that $(u(t), u'(t)), (v(t), v'(t))$ are l.i. at each point $t \in [a, b]$. However $(\alpha, \beta) \cdot (u(a), u'(a)) = 0 = (\alpha, \beta) \cdot (v(a), v'(a))$ says that both $(u(a), u'(a))$ and $(v(a), v'(a))$ are orthogonal to the non-zero vector (α, β) and hence both of these vectors are multiples of the vector $(-\beta, \alpha)$ and hence are l.d., so the latter alternative (that $(u(t), u'(t)), (v(t), v'(t))$ are l.i. at each point $t \in [a, b]$) is impossible.
Note: We used the fact that (α, β) is a non-zero vector; this is true because otherwise the boundary condition in (SL) would say nothing at the end-point $t = a$ and we could find a non-trivial solution of (SL) in case $\lambda = 0$, contradicting the fact that 0 is assumed not to be an eigenvalue of (SL).

5,6,7: Questions 10.1, 10.2, 10.3 from the text.
Solution 10.1: For $a \le x \le y \le b$, we have by the Cauchy-Schwarz inequality (with $\chi_{[x,y]}$ = the indicator function of $[x, y]$) that $|F(x) - F(y)| = |\int_x^y f(t)\,dt| = |(f, \chi_{[x,y]})| \le \|f\|_2 \|\chi_{[x,y]}\|_2 = \|f\|_2 \sqrt{|x-y|}$. On the other hand if $f = \chi_{[a, \frac{1}{2}(a+b)]}$ then we have $F(x) = x - a$ if $x \in [a, \frac{1}{2}(a+b)]$ and $F(x) = \frac{1}{2}(b-a)$ if $x \in [\frac{1}{2}(a+b), b]$, and this function has a corner at $x = \frac{1}{2}(a+b)$, so is not differentiable at $x = \frac{1}{2}(a+b)$.

Solution of 10.2: First note that by a trivial modification of Q2(ii) above, we can check that 0 is not an eigenvalue, so we can follow the procedure of lecture to construct K. For this we must first compute non-trivial solutions u, v of the equation $f'' + \omega^2 f = 0$ with boundary conditions $u'(0) = 0$ and $v'(1) = 0$. Such solutions are given by $u(t) = \cos\omega t$, $v(t) = \cos\omega(t-1)$. For this pair of solutions we have Wronskian $uv' - vu' = -\sin\omega$ which as expected is a non-zero constant because ω is not a multiple of π. Then we know that $K(g)(x) = \frac{1}{\sin\omega}(\varphi(x)u(x) + \psi(x)v(x))$, where $\varphi(x) = \int_x^1 v(\tau)g(\tau)\,d\tau$ and $\psi(x) = \int_0^x u(\tau)g(\tau)\,d\tau$, so $K(g) = \frac{1}{\sin\omega}(\cos\omega x \int_x^1 \cos\omega(\tau-1)g(\tau)\,d\tau + \cos\omega(x-1)\int_0^x \cos\omega\tau\,d\tau)$.

Solution of 10.3: Zero is not an eigenvalue so again we follow lecture: $u(t) = t$, $v(t) = t - 2$ are suitable solutions in this case with Wronskian $uv' - vu' \equiv 2$ and so

$$K(g)(x) = -\tfrac{1}{2}\Big(x \int_x^1 (t-2)g(t)\,dt + (x-2)\int_0^x t g(t)\,dt\Big).$$