Mathematics Department Stanford University
Math 175 Mid-term Test 2008 Solutions

1(a) (3 points): If X is a (real or complex) inner product space, give the proof of the parallelogram identity $\|x-y\|^2+\|x+y\|^2 = 2(\|x\|^2+\|y\|^2)\ \ \forall\, x,y\in X$, assuming that $\|\ \|$ is the inner product norm.

Solution: $\|x-y\|^2 = (x-y,x-y) = \|x\|^2+\|y\|^2-(x,y)-(y,x)$, and since $(y,x)=\overline{(x,y)}$ this gives $\|x-y\|^2 = (x-y,x-y) = \|x\|^2+\|y\|^2-2\operatorname{Re}(x,y)$. Similarly $\|x+y\|^2=(x+y,x+y)=\|x\|^2+\|y\|^2+2\operatorname{Re}(x,y)$ and adding the two identities gives $\|x-y\|^2+\|x+y\|^2=2(\|x\|^2+\|y\|^2)$ as claimed.

(b) (4 points): Let X be a non-trivial finite-dimensional real normed space. Give the proof that the closed ball $B_1=\{x\in X : \|x\|\le 1\}$ is compact. (That is, prove that any sequence $\{x_k\}_{k=1,2,\ldots}\subset B_1$ has a convergent subsequence $\{x_{k_j}\}_{j=1,2,\ldots}$ with $\lim x_{k_j}\in B_1$.)

Hint: You can assume the Bolzano-Weierstrass theorem that every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.

Solution: Let $\|\ \|$ be the norm of X and let $e_1,\ldots,e_n$ be any basis for X. Then $x\in X\Rightarrow x=\sum_{j=1}^n x_je_j$ for some unique $x_1,\ldots,x_n\in\mathbb{R}$. Let $\underline{x}=(x_1,\ldots,x_n)\in\mathbb{R}^n$ and let $\|x\|_e=\|\underline{x}\|_{\mathbb{R}^n}=(\sum_{j=1}^n x_j^2)^{1/2}$. $\|\ \|_e$ is then another norm for X and, from the theorem of lecture that all norms in a finite dimensional normed space are equivalent, we have a constant $C>1$ such that $C^{-1}\|x\|\le\|x\|_e\le C\|x\|$ for all $x\in X$. Now let x_k be any sequence with $\|x_k\|\le 1$; then $\|x_k\|_e=\|\underline{x}_k\|_{\mathbb{R}^n}\le C$ and so by the Bolzano-Weierstrass theorem in $\mathbb{R}^n$ there is a $\underline{y}=(y_1,\ldots,y_n)\in\mathbb{R}^n$ and a subsequence $\underline{x}_{k_j}$ with $\|\underline{x}_{k_j}-\underline{y}\|_{\mathbb{R}^n}\to 0$ and hence $\|x_{k_j}-y\|_e\to 0$, where $y=\sum_{j=1}^n y_je_j$. But then $\|x_{k_j}-y\|\le C\|x_{k_j}-y\|_e\to 0$. Also $\|y\|=\|y-x_{k_j}+x_{k_j}\|\le\|y-x_{k_j}\|+\|x_{k_j}\|\le\|y-x_{k_j}\|+1\to 1$, so $\|y\|\le 1$.

2(a) (4 points): Give the definition of "complete" as applied to a normed linear space. Prove that the linear space ℓ^∞, equipped with the sup norm, is complete. Here ℓ^∞ is the set of bounded real sequences $x=\{x_k\}_{k=1,2,\ldots}$ and the norm is $\|x\|=\sup_k|x_k|$.

Note: You need not prove that ℓ^∞ is a normed linear space—you only have to check completeness.

Solution: A normed linear space X is complete if every Cauchy sequence in X converges with respect to the norm metric.

Let $x^{(n)}$ be a Cauchy sequence in ℓ^∞. Then for each $\varepsilon>0$ there is N such that $\|x^{(n)}-x^{(m)}\|<\varepsilon$ $\forall\, n,m\ge N$, so

$$(*)\qquad |x_k^{(n)}-x_k^{(m)}|<\varepsilon\quad\forall\, n,m\ge N,\ \text{and}\ \forall\, k\ge 1.$$

In particular this means that $\{x_k^{(n)}\}_{n=1,2,\ldots}$ is a Cauchy sequence in $\mathbb{R}$ for each k, and so for each k we have $x_k\in\mathbb{R}$ such that $\lim_{n\to\infty}x_k^{(n)}=x_k$ and using this in $(*)$ (taking limit as $m\to\infty$ in $(*)$ for each k) we have $|x_k^{(n)}-x_k|\le\varepsilon\quad\forall\, n\ge N,\ k\ge 1$ and hence $x^{(n)}-x\in\ell^\infty$ for each $n\ge N$, so $x=x^{(n)}-(x^{(n)}-x)\in\ell^\infty$ and $\|x^{(n)}-x\|\le\varepsilon\quad\forall\, n\ge N$, which is exactly the ε,N definition of $\lim x^{(n)}=x$ with respect to the norm metric.

(b) (4 points): Let $e_1,e_2,\ldots$ be an orthonormal sequence of vectors in a Hilbert space H. Give the proof that $\operatorname{span}\{e_1,e_2,\ldots\}$ dense in H implies that $\lim_{N\to\infty}\|x-\sum_{j=1}^N(x,e_j)e_j\|=0$ for each x.

Hint: Start by proving that $\|x-\sum_{j=1}^N(x,e_j)e_j\|\le\|x-\sum_{j=1}^N a_je_j\|$ for every $x\in H$, every $N\ge 1$, and every choice of scalars $a_1,\ldots,a_N$.

Solution: From lecture/homework we know that $\sum_{j=1}^N(x,e_j)e_j$ is exactly the nearest point projection of x onto $\operatorname{span}\{e_1,\ldots,e_N\}$, so we do indeed have $\|x-\sum_{j=1}^N(x,e_j)e_j\|\le\|x-\sum_{j=1}^N a_je_j\|$ for every $x \in H$, every $N \geq 1$, and every choice of scalars $a_1, \dots, a_N$ as claimed. Let $\varepsilon > 0$ and $x \in H$. Since span$\{e_1, e_2, \dots\}$ is dense we have $\|x - \sum_{j=1}^{N} a_j e_j\| < \varepsilon$ for some $N \geq 1$ and scalars $a_1, \dots, a_N$. Hence by the inequality of the hint (with $M > N$ in place of N and with $a_{N+1} = \cdots = a_M = 0$) we have $\|x - \sum_{j=1}^{M} (x, e_j) e_j\| < \varepsilon$ for all $M > N$.

3(a) (4 points): Let S be a closed linear subspace of a real Hilbert space H and let P_S be the nearest point projection onto S. (Thus $P_S : H \to H$ has the properties that $P_S(H) \subset S$ and $\|x - P_S(x)\| = \min_{y \in S} \|x - y\|$ for each $x \in H$.) Prove that $x - P_S(x) \perp S$ (i.e. $x - P_S(x) \in S^{\perp}$) for each $x \in H$.

Solution: For $y \in S$ and any $\lambda \in \mathbb{R}$ we have $\|x - P_S(x)\|^2 \leq \|x - P_S(x) - \lambda y\|^2 = \|x - P_S(x)\|^2 + |\lambda|^2 \|y\|^2 - 2(x - P_S(x), \lambda y)$ and hence $2(x - P_S(x), \lambda y) \leq |\lambda|^2 \|y\|^2$. Choosing $\lambda = t(x - P_S(x), y)$ where $t \in (0, 1)$ we thus get $2t(x - P_S(x), y)^2 \leq t^2 (x - P_S(x), y)^2 \|y\|^2$, hence $2(x - P_S(x), y)^2 \leq t(x - P_S(x), y)^2 \|y\|^2$, so letting $t \downarrow 0$ gives $(x - P_S(x), y) = 0$ as required.

(b) (4 points) Let ℓ^{∞} be the Banach space of bounded real sequences $x = \{x_k\}_{k=1,2,\dots}$ with the usual sup norm $\|x\| = \sup_k |x_k|$. If $f : \ell^{\infty} \to \mathbb{R}$ is a continuous linear functional, and if e_n is the sequence with 1 in the n'th position and zeros elsewhere, prove that the sequence $\{f(e_k)\}_{k=1,2,\dots}$ is in ℓ^1; that is, prove that $\sum_{k=1}^{\infty} |f(e_k)|$ is a convergent series.

Solution: We are given that f is a bounded linear functional hence $\|f\| = \sup_{\|x\| \leq 1} |f(x)|$ exists as a real number, and $|f(x)| \leq \|f\|$ for every $x \in \ell^{\infty}$ with $\|x\| \leq 1$. Let $a_k = f(e_k)$ and for each $k = 1, \dots, N$ define $x_k = \operatorname{sign} a_k$ (i.e. $+1$ if $a_k > 0$, -1 if $a_k < 0$, and 0 if $a_k = 0$), and let $x_k = 0$ for all $k \geq N + 1$. Then $x = \{x_k\}_{k=1,2,\dots} \in \ell^{\infty}$ with $\|x\| \leq 1$, and so $|f(x)| \leq \|f\|$. On the other hand $|f(x)| = |f(\sum_{k=1}^{N} x_k e_k)| = |\sum_{k=1}^{N} x_k f(e_k)| = \sum_{k=1}^{N} |a_k|$, so we have established $\sum_{k=1}^{N} |f(e_k)| \leq \|f\|$, so $\|f\|$ bounds the partial sums of $\sum_{k=1}^{\infty} |f(e_k)|$, hence the series converges.

4(a) (4 points): Define "Lebesgue measure zero" as applied to a subset $A \subset \mathbb{R}$. Suppose $\varphi_k, k = 1, 2, \dots$ are non-negative step functions on $[0, 1]$ and $\int_{[0,1]} \varphi_k < 2^{-k}$ for each k. Prove that $\varphi_k(x) \to 0$ a.e. $x \in [0, 1]$.
Hint: Consider the step functions $\Phi_N = \sum_{k=1}^{N} \varphi_k$, $N = 1, 2, \dots$.

Solution: A has Lebesgue measure zero if for every $\varepsilon > 0$ there is a sequence $I_1, I_2, \dots$ of open intervals such that $A \subset \cup_{j=1}^{\infty} I_j$ and $\sum_{j=1}^{\infty} |I_j| < \varepsilon$.

Let $\Phi_N = \sum_{k=1}^{N} \varphi_k$. Then Φ_N is a step function and $\Phi_{N+1} \geq \Phi_N$ for each $N \geq 1$. Also $\int_{[0,1]} \Phi_N < \sum_{k=1}^{N} 2^{-k} < 1$, so $\{\Phi_N\}_{N=1,2,\dots}$ is an increasing sequence of step functions with $\{\int_{[0,1]} \Phi_N\}_{N=1,2,\dots}$ bounded, and hence by a theorem of lecture (Theorem 1(i) of the lecture supplement) we know that $\{\Phi_N(x)\}_{N=1,2,\dots}$ is bounded for a.e. $x \in [0, 1]$. That is, a.e. $x \in [0, 1]$ the series $\sum_{k=1}^{\infty} \varphi_k(x)$ (which has non-negative terms) has bounded partial sums and hence is convergent, and hence $\varphi_k(x) \to 0$ a.e. $x \in [0, 1]$.

(b) (3 points): Suppose $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and of class C^1 (i.e. the derivative f' exists and is a continuous function on $\mathbb{R}$). Prove that there is a fixed constant C such that the Fourier coefficients $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$ of f satisfy $|c_n| \leq \frac{C}{|n|}$ for all $n = \pm 1, \pm 2, \dots$.

Solution: $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx$, and by integration by parts we therefore have

$$c_n = -\frac{1}{2\pi i n} f(x) e^{-inx} |_{-\pi}^{\pi} + \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(x) e^{-inx} \, dx = \frac{1}{2\pi i n} \int_{-\pi}^{\pi} f'(x) e^{-inx} \, dx,$$

where we used the fact that $f(x) e^{-inx}$ is a 2π-periodic function, and hence $f(x) e^{-inx} |_{-\pi}^{\pi} = 0$. Thus $|c_n| \leq \frac{1}{2\pi |n|} |\int_{-\pi}^{\pi} f'(x) e^{-inx} \, dx| \leq \frac{1}{2\pi |n|} \int_{-\pi}^{\pi} |f'(x) e^{-inx}| \, dx = \frac{1}{2\pi |n|} \int_{-\pi}^{\pi} |f'(x)| \, dx$, where we used the fact that $|e^{-inx}| = 1$. Since f' is continuous in $[-\pi, \pi]$, its absolute value is bounded by some constant M, and so we have shown that $|c_n| \leq \frac{M}{|n|}$.

Note: The above computation shows that $(f, e^{inx}) = \frac{1}{in} (f', e^{inx})$ for a C^1 function f as above.