# MATH 103 SAMPLE FINAL SOLUTIONS

1. Consider the data points $(-1, 15), (0, 8), (1, 5), (2, 0)$.

   (a) Find the least-squares line for the data points above.
   **Solution.** Let $f(x) = c_0 + c_1 x$. Then $c_0$ and $c_1$ must satisfy

$$\begin{array}{ccccc} c_0 & - & c_1 & = & 15 \\ c_0 & & & = & 8 \\ c_0 & + & c_1 & = & 5 \\ c_0 & + & 2c_1 & = & 0 \end{array}$$

   This system is inconsistent. Since

$$A^T A = \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \qquad A^T \vec{b} = \begin{bmatrix} 28 \\ -10 \end{bmatrix}$$

   the augmented matrix for the normal equations is

$$\left[\begin{array}{cc|c} 4 & 2 & 28 \\ 2 & 6 & -10 \end{array}\right]$$

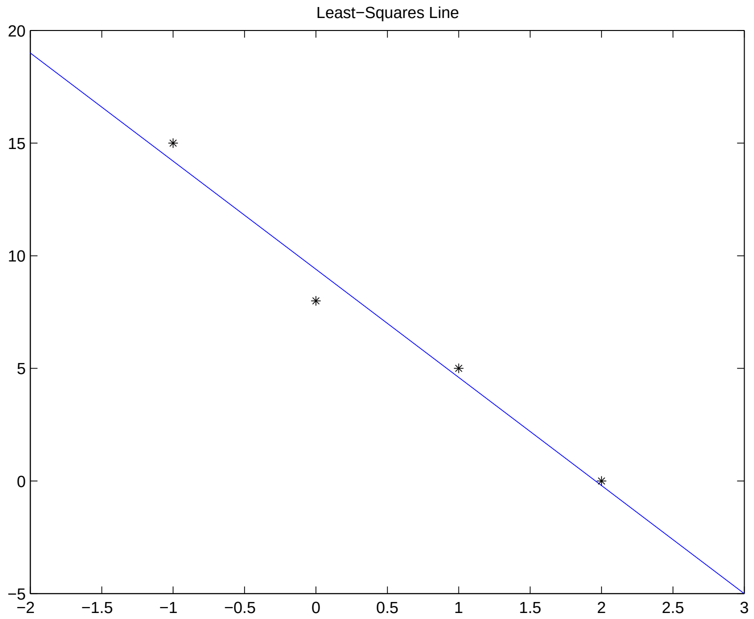
   Its reduced row echelon form is

$$\left[\begin{array}{cc|c} 1 & 0 & 47/5 \\ 0 & 1 & -24/5 \end{array}\right]$$

   Thus the least-squares line is

$$\boxed{f(x) = \frac{47}{5} - \frac{24}{5}x}$$

   (b) Plot the points and the least-squares approximation.
   **Solution.**



(c) Find a cubic polynomial of the form $f(x) = c_0 + c_1x + c_2x^2 + c_3x^3$ which passes through all four points.

**Solution.** The coefficients must satisfy the system $A\vec{x} = \vec{b}$ where

$$A = \begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \end{bmatrix} \qquad \vec{b} = \begin{bmatrix} 15 \\ 8 \\ 5 \\ 0 \end{bmatrix}$$

The solution is

$$\vec{x} = \begin{bmatrix} 8 \\ -4 \\ 2 \\ -1 \end{bmatrix}$$

so $\boxed{f(x) = 8 - 4x + 2x^2 - x^3}$.

2. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}.$$

**Solution.** Call these vectors $\vec{v}_1, \vec{v}_2$ and $\vec{v}_3$. Normalizing the first vector gives

$$\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Next,

$$\vec{v}_2 - (\vec{v}_2 \cdot \vec{w}_1)\vec{w}_1 = \vec{v}_2 - 4\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

so normalizing gives

$$\vec{w}_2 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

Finally

$$\vec{v}_3 - (\vec{v}_3 \cdot \vec{w}_1)\vec{w}_1 - (\vec{v}_3 \cdot \vec{w}_2)\vec{w}_2 = \vec{v}_3 - 6\vec{w}_1 - 4\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ -2 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

so normalizing gives

$$\vec{w}_3 = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

3. Let $V$ be any subspace of $\mathbb{R}^n$. Let $P_1$ denote the matrix for the orthogonal projection onto $V$ and let $P_2$ denote the matrix for the orthogonal projection onto $V^\perp$.

   (a) Show that $P_1$ and $P_2$ are symmetric. Hint: Use Fact 5.3.10.

   **Solution.** By Fact 5.3.10, $P_1$ takes the form $AA^T$ for some matrix $A$. Thus $P^T = (AA^T)^T = (A^T)^T A^T = AA^T = P_1$, so $P_1$ is symmetric. Similarly $P_2$ is symmetric.

   (b) Show that $P_1 P_2 = P_2 P_1 = 0$ (the zero matrix). Hint: Explain why the columns (and rows) of $P_1$ are vectors in $V$. Where are the columns and rows of $P_2$?

   **Solution.** The image of $P_1$ must be $V$. But the image of $P_1$ is the span of the columns of $P_1$. Thus each column of $P_1$ is a vector in $V$, and since $P_1$ is symmetric by part (a), the rows of $P_1$ are also vectors in $V$. Likewise the rows and columns of $P_2$ are vectors in $V^\perp$. Each entry in $P_1 P_2$ is the dot product of a row of $P_1$ with a column of $P_2$, so each entry is the dot product of a vector in $V$ with a vector in $V^\perp$. By definition of $V^\perp$, all such dot products are zero. Hence $P_1 P_2 = 0$, and similarly $P_2 P_1 = 0$.

   (c) Show that $P_1 + P_2 = I_n$. Hint: For any $\vec{x} \in \mathbb{R}^n$, consider $\vec{w} = \vec{x} - P_1\vec{x}$. Where is $\vec{w}$? Now apply $P_2$ to both sides.

   **Solution.** Let $\vec{x} \in \mathbb{R}^n$. By Fact 5.1.6, the projection of $\vec{x}$ onto $V$ is the unique vector $\vec{v}$ such that $\vec{x} - \vec{v}$ is in $V^\perp$. Since $\vec{v} = P_1\vec{x}$ this means that

   $$\vec{w} = \vec{x} - P_1\vec{x}$$

   is in $V^\perp$. Applying $P_2$ to both sides gives

   $$P_2\vec{w} = P_2\vec{x} - P_2 P_1 \vec{x} = P_2\vec{x}$$

   But since $\vec{w}$ is in $V^\perp$, $P_2\vec{w} = \vec{w}$, so $\vec{w} = P_2\vec{x}$. Thus

   $$\vec{x} = P_1\vec{x} + P_2\vec{x} = (P_1 + P_2)\vec{x}$$

   for all $\vec{x} \in \mathbb{R}^n$. This means that $P_1 + P_2$ is the matrix for the identity transformation, i.e. $P_1 + P_2 = I_n$.

4. Let $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$.

   (a) Find a basis for $V^\perp$.

   **Solution.**

   $$V = \left(\text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)\right)^\perp \qquad \Longrightarrow \qquad V^\perp = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}\right)$$

(b) Find the matrix (in standard coordinates) for the orthogonal projection onto $V^\perp$.

**Solution.** Since $V^\perp$ is spanned by the unit vector

$$\vec{u} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

the matrix for the orthogonal projection onto $V^\perp$ is (by Fact 5.3.10)

$$P_2 = \vec{u}\vec{u}^T = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

(c) Find the matrix (in standard coordinates) for the orthogonal projection onto $V$. Hint: Use Question 3c.

**Solution.** By Question 3c, the matrix for the orthogonal projection onto $V$ is

$$P_1 = I_4 - P_2 = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{bmatrix}$$

5. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

(a) Find all eigenvalues of $A$ and a basis for the eigenspace associated with each eigenvalue.

**Solution.** The characteristic polynomial is $f_A(\lambda) = (\lambda - 1)(\lambda - 2)(\lambda + 1)$, so the eigenvalues are $\lambda = 1, 2, -1$. The eigenspaces are

$$E_1 = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right) \quad E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) \quad E_{-1} = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}\right)$$

(b) Diagonalize $A$. That is, find an invertible matrix $S$ and a diagonal matrix $D$ such that $S^{-1}AS = D$.

**Solution.**

$$S = \begin{bmatrix} -1 & 1 & 1 \\ 0 & 1 & -2 \\ 1 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(c) Solve the system $\vec{x}(t+1) = A\vec{x}(t)$ with initial data $\vec{x}_0 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$.

**Solution.** Since

$$\begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} = -3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

the solution is

$$\vec{x}(t) = -3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + 2(2)^t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (-1)^t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

6. Consider the plane $V = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$.

(a) Find a basis $\{\vec{v}_1, \vec{v}_2\}$ for $V$ and a basis $\{\vec{v}_3\}$ for $V^\perp$.

**Solution.**

$$V = \operatorname{Span}\left( \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right) \qquad V^\perp = \operatorname{Span}\left( \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

(b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by reflection through $V$. That is, $T$ sends each vector in $\mathbb{R}^3$ to its mirror image on the opposite side of $V$. Write down the matrix $B$ for $T$ with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$.

**Solution.**

$$\begin{array}{lclclclcl} T(\vec{v}_1) & = & \vec{v}_1 & = & 1\vec{v}_1 & + & 0\vec{v}_2 & + & 0\vec{v}_3 \\ T(\vec{v}_2) & = & \vec{v}_2 & = & 0\vec{v}_1 & + & 1\vec{v}_2 & + & 0\vec{v}_3 \\ T(\vec{v}_3) & = & -\vec{v}_1 & = & 0\vec{v}_1 & + & 0\vec{v}_2 & + & (-1)\vec{v}_3 \end{array}$$

Thus

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(c) Write down the matrix $A$ for $T$ with respect to the standard basis for $\mathbb{R}^3$.

**Solution.** Let

$$S = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Then

$$A = SBS^{-1} = \frac{1}{3} \begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}$$

7. Find an example of each of the following. Hint: There exist 2 by 2 examples in each case.

(a) A matrix which is diagonalizable but not invertible.

**Solution.**

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) A matrix which is invertible but not diagonalizable.

**Solution.**

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

(c) Diagonalizable matrices $A$ and $B$ such that $A + B$ is not diagonalizable.

**Solution.**

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

(d) Diagonalizable matrices $A$ and $B$ such that $AB$ is not diagonalizable.

**Solution.**

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

(e) Symmetric matrices $A$ and $B$ such that $AB$ is not symmetric.

**Solution.**

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

8. Suppose $A$ is a symmetric matrix and $B = A^2 + I_n$. Show that $B$ is invertible. Hint: What are the possible eigenvalues of $B$?

**Solution.** By Fact 8.1.1 $A$ has an eigenbasis $\{\vec{v}_1, \dots, \vec{v}_n\}$ with *real* (not necessarily distinct) eigenvalues $\lambda_1, \dots, \lambda_n$. For each $i = 1, \dots, n$ we have

$$B\vec{v}_i = (A^2 + I_n)\vec{v}_i = A^2\vec{v}_i + \vec{v}_i = (\lambda_i^2 + 1)\vec{v}_i$$

Hence $\{\vec{v}_1, \dots, \vec{v}_n\}$ is an eigenbasis of $B$ with *positive* eigenvalues $\lambda_1^2 + 1, \dots, \lambda_n^2 + 1$. Thus $B = SDS^{-1}$ where $S$ has the eigenbasis as its columns and $D$ is the diagonal matrix with $\lambda_1^2 + 1, \dots, \lambda_n^2 + 1$ on the diagonal. Since all of these diagonal entries are positive, $D$ is invertible, so $B$ must be invertible.