MATH 103 SAMPLE FINAL

1. Consider the data points $(-1, 15), (0, 8), (1, 5), (2, 0)$.

   (a) Find the least-squares line for the data points above.

   (b) Plot the points and the least-squares approximation.

   (c) Find a cubic polynomial of the form $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ which passes through all four points.

2. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by the following vectors:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 1 \\ 3 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \end{bmatrix}.$$

3. Let V be any subspace of $\mathbb{R}^n$. Let P_1 denote the matrix for the orthogonal projection onto V and let P_2 denote the matrix for the orthogonal projection onto $V^\perp$.

   (a) Show that P_1 and P_2 are symmetric. Hint: Use Fact 5.3.10.

   (b) Show that $P_1 P_2 = P_2 P_1 = 0$ (the zero matrix). Hint: Explain why the columns (and rows) of P_1 are vectors in V. Where are the columns and rows of P_2?

   (c) Show that $P_1 + P_2 = I_n$. Hint: For any $\vec{x} \in \mathbb{R}^n$, consider $\vec{w} = \vec{x} - P_1 \vec{x}$. Where is $\vec{w}$? Now apply P_2 to both sides.

4. Let $V = \{\vec{x} \in \mathbb{R}^4 \mid x_1 + x_2 + x_3 + x_4 = 0\}$.

   (a) Find a basis for $V^\perp$.

   (b) Find the matrix (in standard coordinates) for the orthogonal projection onto $V^\perp$.

   (c) Find the matrix (in standard coordinates) for the orthogonal projection onto V. Hint: Use Question 3c.

5. Let

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

   (a) Find all eigenvalues of A and a basis for the eigenspace associated with each eigenvalue.

   (b) Diagonalize A. That is, find an invertible matrix S and a diagonal matrix D such that $S^{-1} A S = D$.

   (c) Solve the system $\vec{x}(t+1) = A\vec{x}(t)$ with initial data $\vec{x}_0 = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix}$.

6. Consider the plane $V = \{\vec{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\}$.

   (a) Find a basis $\{\vec{v}_1, \vec{v}_2\}$ for V and a basis $\{\vec{v}_3\}$ for $V^\perp$.

   (b) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by reflection through V. That is, T sends each vector in $\mathbb{R}^3$ to its mirror image on the opposite side of V. Write down the matrix B for T with respect to the basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ of $\mathbb{R}^3$.

   (c) Write down the matrix A for T with respect to the standard basis for $\mathbb{R}^3$.

7. Find an example of each of the following. Hint: There exist 2 by 2 examples in each case.

   (a) A matrix which is diagonalizable but not invertible.

   (b) A matrix which is invertible but not diagonalizable.

   (c) Diagonalizable matrices A and B such that $A + B$ is not diagonalizable.

   (d) Diagonalizable matrices A and B such that AB is not diagonalizable.

   (e) Symmetric matrices A and B such that AB is not symmetric.

8. Suppose A is a symmetric matrix and $B = A^2 + I$. Show that B is invertible. Hint: What are the possible eigenvalues of B?