Math 103 Final

Tuesday March 20^{th}, 2007

Time: 3 hours
Total: 160 points

This is a closed book test. Calculators and other computational aids are strictly forbidden. Lucky charms won't help you, but feel free to use them. Cell phones won't help you either, but do not feel free to use them. Good luck ☺

1. The aim of this question is to connect two curves 'smoothly'. Let $\ell_1 = \{(x,y) \mid x < 0, y = 0\}$ and $\ell_2 = \{(x,y) \mid x > 1, y = 1\}$. In words, ℓ_1 and ℓ_2 are two horizontal line segments. ℓ_1 starts at the origin and goes left (horizontally). ℓ_2 starts at $(1,1)$ and goes right (horizontally).

   [10] (a) Let $c_0, c_1, c_2, c_3 \in \mathbb{R}$ and define $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$. Find $c_0, \dots c_3$ so that $f(0) = 0$, $f(1) = 1$, $f'(0) = 0$ and $f'(1) = 0$. [If you draw the graph of $y = f(x)$, this will give you a curve that 'smoothly' joins ℓ_1 and ℓ_2. For those who 'forgot' calculus, $f'(x) = c_1 + 2c_2 x + 3c_3 x^2$.]

   [10] (b) Let $c_0, c_1, c_2 \in \mathbb{R}$, and define $g(x) = c_0 + c_1 x + c_2 x^2$. Show that it is *not* possible to find c_0, c_1, c_2 so that $g(0) = 0$, $g(1) = 1$, $g'(0) = 0$ and $g'(1) = 0$.

   [10] (c) Suppose you wanted to smoothly join ℓ_1 and ℓ_2 using a function g as defined in the previous subpart. As you've shown above, an *exact* solution is not possible. Explain what the next best thing to do would be. Write down an explicit expression for c_0, c_1, c_2 obtained in this manner. [Your answer can involve products and inverses of matrices, as long as the matrices consist of only numbers (i.e. no variables).]

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. Let $\vec{v}_1 = \left(\begin{smallmatrix} \cos\alpha \\ \sin\alpha \end{smallmatrix}\right)$ and $\vec{v}_2 = \left(\begin{smallmatrix} -\sin\alpha \\ \cos\alpha \end{smallmatrix}\right)$. Suppose $T\vec{v}_1 = \vec{v}_1$ and $T\vec{v}_2 = -\vec{v}_2$.

   [10] (a) Find the matrix of T and T^{-1} in basis $\{\vec{v}_1, \vec{v}_2\}$.

   [10] (b) Find the matrix of T and T^{-1} in standard basis.

   [5] (c) Does the transformation T have any physical meaning?

3. [5] (a) Define what it means for $\{\vec{v}_1, \dots \vec{v}_k\}$ to be linearly independent.

   [10] (b) Suppose $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent. Show that $\{\vec{v}_1, 2\vec{v}_2, 3\vec{v}_3\}$ is linearly independent.

4. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Suppose $ad - bc = -1$ and $A^{\text{t}} = A^{-1}$.

   [5] (a) Write down a formula for A^{-1}. [Your answer can involve a, b, c, d.]

   [10] (b) Show that $a + d = 0$.

   [10] (c) Compute the characteristic polynomial of A. [Your answer should *not* involve a, b, c or d.]

   [5] (d) What are the eigenvalues of A?

5. [10] Let $V \subseteq \mathbb{R}^4$ be defined by

$$V = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \\ 0 \end{pmatrix} \right\}$$

   Let $\vec{x} \in \mathbb{R}^4$. Find a matrix P so that $P\vec{x}$ is the orthogonal projection of $\vec{x}$ onto V.

6. [10] Find a matrix P such that

$$P^{-1} \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{pmatrix} P = \begin{pmatrix} 3 & & & \\ & 1 & & \\ & & 4 & \\ & & & 2 \end{pmatrix}$$

   [Hint: If you attempt doing this by trial and error, you will waste 3 hours and get nowhere. The hint is to remember conceptually how $P^{-1}AP$ relates to A.]

10 7. Let A be an $n \times n$ matrix. Suppose n is odd, and $A^{\mathrm{t}} = -A$, then show that A is not invertible. [NOTE: When n is even, A could be invertible. If your proof works even though n is even, you've certainly messed up]

8. Let P be an $n \times n$ *symmetric* matrix such that $P^2 = P$.

10 (a) If $\vec{x}, \vec{y} \in \mathbb{R}^n$, show that $\vec{x} \cdot (P\vec{y}) = (P\vec{x}) \cdot \vec{y}$. [HINT: This has nothing to do with $P^2 = P$.]

10 (b) Let $U = \ker P$ and $V = \operatorname{im} P$. Show that $U = V^{\perp}$. [HINT: This has nothing to do with $P^2 = P$.]

10 (c) Show that P is the matrix of the orthogonal projection onto the subspace V. [HINT: This hint is not the same as the previous two hints.]