Math 103 Midterm

Tuesday Febuary 6^{th}, 2007

Time: 75 mins
Total: 75 points

This is a closed book test. Calculators and other computational aids are strictly forbidden. Lucky charms won't help you, but feel free to use them. Cell phones won't help you either, but do not feel free to use them. Good luck ⌣̈

$\boxed{5}$ 1. Let M be an 3×3 matrix. Suppose

$$\text{r.e.}\left(\begin{array}{ccc|c} & & & 1 \\ & M & & 0 \\ & & & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & & & x_1 \\ & 1 & & x_2 \\ & & 1 & x_3 \end{array}\right).$$

That is, suppose that the row echeleon form of the augmented 3×4 matrix on the left is the matrix on the right. Let $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$. What is $M\vec{x}$?

Solution. To solve the linear system of equations $M\vec{x} = \vec{e}_1$, you would row reduce the augmented matrix (on the left), and read off the answer from the right. Thus $M\vec{x} = \vec{e}_1$. □

$\boxed{10}$ 2. Let $B = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$. Find $\ker(B)$ and $\operatorname{im}(B)$ and express them as the span of linearly independent vectors.

Solution. We know $\operatorname{im}(B) = \operatorname{span}\{\left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)\} = \operatorname{span}\{\vec{e}_1, \vec{e}_2\} = \mathbb{R}^2$. To find the kernel, the row reduced echeleon form of B is $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{smallmatrix}\right)$. Thus $\ker B$ consists of vectors $\vec{x}$ where $x_1 = -x_3$ and $x_2 = x_3$. So $\ker(B) = \operatorname{span}\{-\vec{e}_1 + \vec{e}_2 + \vec{e}_3\}$. □

3. Let $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$ be the standard basis vectors of $\mathbb{R}^3$. Let $\vec{v}_1 = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$, $\vec{v}_2 = \vec{e}_1 - \vec{e}_2$ and $\vec{v}_3 = \vec{e}_1 + \vec{e}_2 - 2\vec{e}_3$.

$\boxed{5}$ (a) Let A be the matrix defined by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}.$$

Compute the rank of A, and the inverse of A (if the inverse exists).

Solution. The row reduction trick gives

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & -\frac{1}{2} & \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \end{pmatrix}$$

Since A has an inverse, it must have full rank thus $\operatorname{rank} A = 3$. □

$\boxed{5}$ (b) Are $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent? Justify.

Solution. Yes. Note that the columns of A are exactly $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$. The rank of A is the number of linearly independent rows, and is also equal to the number of linearly independent columns. □

$\boxed{5}$ (c) Let $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$. Find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$.

Solution. Note $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$ means

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = A \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Hence

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = A^{-1}\vec{x}$$

Explicitly, $\alpha_1 = \frac{1}{3}(x_1 + x_2 + x_3)$, $\alpha_2 = \frac{1}{2}(x_1 - x_2)$, and $\alpha_3 = \frac{1}{6}(x_1 + x_2 - 2x_3)$. □

$\boxed{5}$ (d) If $\alpha_1, \alpha_2, \alpha_3$ and $\vec{x}$ are as in the previous part, find a matrix B such that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Solution. Freebie. If you did the previous subpart by some other method, then you have to do some work here. If not, just say $B = A^{-1}$. □

$\boxed{5}$ (e) What is $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? Justify.

Solution. Part (c) says exactly that $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \mathbb{R}^3$. □

$\boxed{5}$ (f) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose we know $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = \frac{1}{\sqrt{2}}\vec{v}_3$, $T(\vec{v}_3) = -\sqrt{2}\vec{v}_2$. Find the matrix of the linear transformation T. [The transformation T turns out to be a rotation on $\mathbb{R}^3$. As you can see here, three dimensional rotations are a little more complicated than two dimensional ones.]

Solution. If you tried to do this explicitly, you will run out of time. It leads to a system of 9 equations with 9 variables, which is quite painful to solve. The short method is as follows: Let $\vec{x} \in \mathbb{R}^3$, and $\alpha_1, \alpha_2, \alpha_3$ be such that $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$. Then $T(\vec{x}) = \alpha_1\vec{v}_1 + \frac{1}{\sqrt{2}}\alpha_2\vec{v}_2 - \sqrt{2}\alpha_3\vec{v}_3$. In matrix form,

$$T(\vec{x}) = A \begin{pmatrix} \alpha_1 \\ \frac{1}{\sqrt{2}}\alpha_3 \\ -\sqrt{2}\alpha_2 \end{pmatrix} = A \begin{pmatrix} 1 & & \\ & & \sqrt{2} \\ & \frac{1}{\sqrt{2}} & \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$$

Now using part (d) we get

$$T(\vec{x}) = A \begin{pmatrix} 1 & & \\ & & \sqrt{2} \\ & \frac{1}{\sqrt{2}} & \end{pmatrix} A^{-1}\vec{x}$$

Normally you should explicitly compute the matrix product above. But given the time constraints of the midterm if you leave it as above, I won't mind too much. □

$\boxed{5}$ 4. (a) Give an example of three (non-zero) vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$ and six real numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ such that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$, however $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$ and $\alpha_3 \neq \beta_3$.

Solution. Pick $\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = \vec{e}_1$, and $\alpha_1 = 1, \alpha_2 = 1, \alpha_3 = -2$. □

$\boxed{5}$ (b) If the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent, and there exist six real numbers $\alpha_1, \alpha_2, \alpha_3$, $\beta_1, \beta_2, \beta_3$ such that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$, then show that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$ and $\alpha_3 = \beta_3$.

Solution. $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$ implies $(\alpha_1 - \beta_1)\vec{v}_1 + (\alpha_2 - \beta_2)\vec{v}_2 + (\alpha_3 - \beta_3)\vec{v}_3 = 0$, and by definition of linear independence $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$, $\alpha_3 = \beta_3$. □

$\boxed{10}$ 5. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Suppose $ad \neq 0$, and $ad - bc \neq 0$. Use the row reduction trick from the book / class to find the inverse of the matrix A. If $ad = 0$ (but $ad - bc \neq 0$), does something go wrong when row reducing?

Solution. We row reduce as follows

$$\left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & \frac{ad-bc}{a} & -\frac{c}{a} & 1 \end{array}\right) \to$$
$$\to \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

Thus $A^{-1} = \frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. If $ad = 0$, and $ad - bc \neq 0$, then we note that neither b nor c can be 0. Thus we start by swapping the first two rows, and then row reduce as above. $\square$

$\boxed{10}$ 6. Show that any $m+1$ vectors in $\mathbb{R}^m$ are linearly dependent. [Feel free to use without proof any thing said in class or homework. Statements from the book can be used as long as they don't involve the concept of 'dimension', and are not exactly the same as the statement of this question.]

Solution. Any solution of this problem that used 'dimension', or the fact that any m linearly independent vectors in $\mathbb{R}^m$ span $\mathbb{R}^m$ was only given partial credit. The reason being that both these facts rely on this problem, so your proof is 'circular'. (Depending on how correctly and well you wrote your proof, you probably got 8 or 9 points in this case. If your proof was fundamentally incorrect, you got 4 or lower). The point of this problem is to show you that the above statement reduces to solving m linear homogeneous equations in $m+1$ variables.

A non-circular proof is as follows: Pick $\vec{v}_1, \dots, \vec{v}_{m+1} \in \mathbb{R}^m$. Let M be the matrix with columns $\vec{v}_1, \dots \vec{v}_m$. Note that the system of equations $M\vec{x} = 0$ is a system of $m+1$ variables, and m equations (since M is an $m \times (m+1)$ matrix). From your first and second homeworks we know that this system has infinitely many solutions. In particular it has at least one non-zero solution, say $\alpha_1, \dots \alpha_{m+1}$. Thus $\alpha_1\vec{v}_1 + \cdots + \alpha_{m+1}\vec{v}_{m+1} = 0$, where not all α_i are 0. $\square$