Math 103 Midterm

Tuesday Febuary 6^{th}, 2007

Time: 75 mins
Total: 75 points

This is a closed book test. Calculators and other computational aids are strictly forbidden. Lucky charms won't help you, but feel free to use them. Cell phones won't help you either, but do not feel free to use them. Good luck ☺

[5] 1. Let M be an 3×3 matrix. Suppose

$$\text{r.e.}\left(\begin{array}{ccc|c} & & & 1 \\ & M & & 0 \\ & & & 0 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & & & x_1 \\ & 1 & & x_2 \\ & & 1 & x_3 \end{array}\right).$$

That is, suppose that the row echeleon form of the augmented 3×4 matrix on the left is the matrix on the right. Let $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$. What is $M\vec{x}$?

[10] 2. Let $B = \left(\begin{smallmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \end{smallmatrix}\right)$. Find $\ker(B)$ and $\operatorname{im}(B)$ and express them as the span of linearly independent vectors.

3. Let $\vec{e}_1, \vec{e}_2, \vec{e}_3$ be the standard basis vectors of $\mathbb{R}^3$. Let $\vec{v}_1 = \vec{e}_1+\vec{e}_2+\vec{e}_3$, $\vec{v}_2 = \vec{e}_1-\vec{e}_2$ and $\vec{v}_3 = \vec{e}_1+\vec{e}_2-2\vec{e}_3$.

[5] (a) Let A be the matrix defined by

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{pmatrix}.$$

Compute the rank of A, and the inverse of A (if the inverse exists).

[5] (b) Are $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent? Justify.

[5] (c) Let $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3$. Find $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3$.

[5] (d) If $\alpha_1, \alpha_2, \alpha_3$ and $\vec{x}$ are as in the previous part, find a matrix B such that

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = B \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

[5] (e) What is $\operatorname{span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$? Justify.

[5] (f) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Suppose we know $T(\vec{v}_1) = \vec{v}_1$, $T(\vec{v}_2) = \frac{1}{\sqrt{2}}\vec{v}_3$, $T(\vec{v}_3) = -\sqrt{2}\vec{v}_2$. Find the matrix of the linear transformation T. [The transformation T turns out to be a rotation on $\mathbb{R}^3$. As you can see here, three dimensional rotations are a little more complicated than two dimensional ones.]

[5] 4. (a) Give an example of three (non-zero) vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in \mathbb{R}^4$ and six real numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ such that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$, however $\alpha_1 \neq \beta_1$, $\alpha_2 \neq \beta_2$ and $\alpha_3 \neq \beta_3$.

[5] (b) If the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ are linearly independent, and there exist six real numbers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ such that $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \beta_3\vec{v}_3$, then show that $\alpha_1 = \beta_1$, $\alpha_2 = \beta_2$ and $\alpha_3 = \beta_3$.

[10] 5. Let $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Suppose $ad \neq 0$, and $ad - bc \neq 0$. Use the row reduction trick from the book / class to find the inverse of the matrix A. If $ad = 0$ (but $ad - bc \neq 0$), does something go wrong when row reducing?

[10] 6. Show that any $m + 1$ vectors in $\mathbb{R}^m$ are linearly dependent. [Feel free to use without proof any thing said in class or homework. Statements from the book can be used as long as they don't involve the concept of 'dimension', and are not exactly the same as the statement of this question.]