Basics of proof theory

Ling 233B 1/16/02

1. Curry Howard Isomorphism
   - Proofs as first-class objects
   - Mechanism by which glue derivations assemble meaning terms
2. Linear Logic
   - Motivated by proof theoretic considerations

What You Need to Remember

Linear Logic	
Traditional:	Linear:
$A, A \to B \vdash B$	$A, A \multimap B \vdash B$
$A, A \to B \vdash A \wedge B$	$A, A \multimap B \nvdash A \otimes B$
Re-use A	Cannot re-use A
$A, B \vdash A$	$A, B \nvdash A$
Discard B	Cannot discard B

$$A \multimap (B \multimap C) \equiv$$
$$B \multimap (A \multimap C) \equiv$$
$$(A \otimes B) \multimap C$$

Curry Howard

$$\frac{a : A \qquad P : A \multimap B}{P(a) : B} \multimap_{\mathcal{E}}$$

$$\frac{\begin{array}{c}[x : A]^i \\ \vdots \\ P(x) : B\end{array}}{\lambda x P(x) : A \multimap B} \multimap_{\mathcal{I},i}$$

Overview: Curry-Howard Isomorphism

- Motivation: proofs are interesting in their own right
- Inference rules and λ-operations
- Proof normalization and λ-reduction
- Limitations of the CHI
  Natural deduction, constructive logics

Proofs as First-Class Objects

Proofs as the senses of formulas:

— instead of "*When is A true?*", ask "*What is a proof of A*"?

Problem: we have no direct access to proofs

— only to their syntactic representations as derivations in some proof system

— syntax of derivations can introduce spurious distinctions

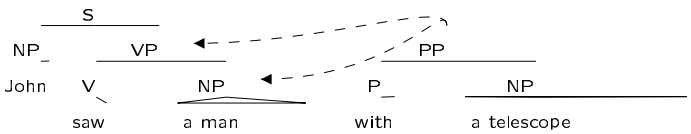
Aim: find the correct identity criteria for proofs

— "*No entity without identity*"

Esoteric, logician's concern? No.

Parsing as Deduction

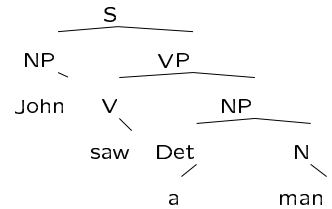
Context Free Grammar				Deductive Grammar				
S	$\Rightarrow$	NP	VP	${}_i\mathrm{NP}_j$	$\otimes$	${}_j\mathrm{VP}_k$	$\multimap$	${}_i\mathrm{S}_k$
VP	$\Rightarrow$	V	NP	${}_i\mathrm{V}_j$	$\otimes$	${}_j\mathrm{VP}_k$	$\multimap$	${}_i\mathrm{VP}_k$
PP	$\Rightarrow$	P	NP	${}_i\mathrm{P}_j$	$\otimes$	${}_j\mathrm{NP}_k$	$\multimap$	${}_i\mathrm{PP}_k$
NP	$\Rightarrow$	NP	PP	${}_i\mathrm{NP}_j$	$\otimes$	${}_j\mathrm{PP}_k$	$\multimap$	${}_i\mathrm{NP}_k$
VP	$\Rightarrow$	VP	PP	${}_i\mathrm{VP}_j$	$\otimes$	${}_j\mathrm{PP}_k$	$\multimap$	${}_i\mathrm{VP}_k$
NP	$\Rightarrow$	Det	N	${}_i\mathrm{Det}_j$	$\otimes$	${}_j\mathrm{N}_k$	$\multimap$	${}_i\mathrm{NP}_k$

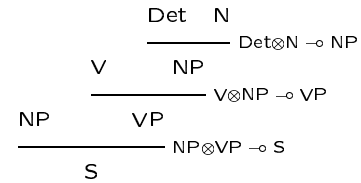


Corresponds to two distinct proofs of:

${}_0\mathrm{NP}_1$,	${}_1\mathrm{V}_2$,	${}_2\mathrm{Det}_3$,	${}_3\mathrm{N}_4$,	${}_4\mathrm{P}_5$,	${}_5\mathrm{Det}_6$,	${}_6\mathrm{N}_7$	$\vdash {}_0\mathrm{S}_7$
John	*saw*	*a*	*man*	*with*	*a*	*telescope*	

Parse Trees as Proof Trees





Distinct proofs/parses have distinct meanings

Curry-Howard Isomorphism

Semantics of Proofs: Implication Elimination as Functional Application

Natural deduction rule for (intuitionistic) implication elimination:

$$\frac{A \to B \qquad A}{B} \to_{\mathcal{E}}$$

$A \to B$: function f that takes a proof a of A to give a proof $f(a)$ of B

$$\frac{f : A \to B \qquad a : A}{f(a) : B} \to_{\mathcal{E}}$$

(Also works for linear implication, $\multimap$)

Implication Introduction as Lambda Abstraction

Natural deduction rule for implication introduction

$$\dfrac{\begin{array}{c}[A]^i \\ \vdots \\ B\end{array}}{A \to B} \to_{\mathcal{I}}, i$$

Assuming A allows one to prove B.

Therefore, discharging the assumption, $[A]^i$, one proves $A \to B$

With proof terms

$$\dfrac{\begin{array}{c}[x : A]^i \\ \vdots \\ P : B\end{array}}{\lambda x.P : A \to B} \to_{\mathcal{I}}, i$$

Identity Criteria for Proofs

Two 'proofs' of $A, A \to B \vdash B$:

$$\dfrac{A \to B \qquad A}{B} \to_{\mathcal{E}} \qquad\qquad \dfrac{\dfrac{\dfrac{[A]^1 \qquad A \to B}{B} \to_{\mathcal{E}}}{A \to B} \to_{\mathcal{I}}, 1 \qquad A}{B} \to_{\mathcal{E}}$$

These are not really distinct proofs:

Lambda-Equivalence of Proof Terms

Include proof terms in previous derivations:

$$\dfrac{f : A \to B \qquad a : A}{f(a) : B} \to_{\mathcal{E}} \qquad\qquad \dfrac{\dfrac{\dfrac{[x : A]^1 \qquad f : A \to B}{f(x) : B} \to_{\mathcal{E}}}{\lambda x.f(x) : A \to B} \to_{\mathcal{I}}, 1 \qquad a : A}{(\lambda x.f(x))(a) : B} \to_{\mathcal{E}}$$

Note: $f(a) = (\lambda x.f(x))(a)$

λ-equivalence of proof terms: semantic identity of derivations.

Curry-Howard Isomorphism (CHI)

CHI = Pairing of proof rules with λ-operations on proof terms

- But doesn't work for all logics, or proof systems

Intimate relation between logic and type-theory.

- Normalization of proofs isomorphic to λ-reduction of proof terms

Defines interesting identity criteria for proofs

- Syntactically distinct derivations corresponding to same proof

Proof Normalization

Natural Deduction

Each connective defined by paired introduction & elimination rules

Introduction

$$\dfrac{A \qquad B}{A \wedge B}\,\wedge_{\mathcal{I}}$$

$$\dfrac{\begin{array}{c}[A]^i\\ \vdots\\ B\end{array}}{A \to B}\,\to_{\mathcal{I},i}$$

Elimination

$$\dfrac{A \wedge B}{A}\,\wedge_{\mathcal{E}} \qquad \dfrac{A \wedge B}{B}\,\wedge_{\mathcal{E}}$$

$$\dfrac{A \qquad A \to B}{B}\,\to_{\mathcal{E}}$$

Detours in Proofs

Two types of detour

β: introduction of a connective immediately followed by its elimination

η: elimination of a connective immediately followed by its introduction

η and β detour

$$\dfrac{\dfrac{\dfrac{[A]^1 \qquad A \to B}{B}\,\to_{\mathcal{E}}}{A \to B}\,\to_{\mathcal{I}},1 \qquad A}{B}\,\to_{\mathcal{E}}$$

β detour

$$\dfrac{\dfrac{A \qquad B}{A \wedge B}\,\wedge_{\mathcal{I}}}{A}\,\wedge_{\mathcal{E}}$$

Normalization:

Remove detours by β/η-normalization rules.

Some Normalization Rules

$$\dfrac{\dfrac{\begin{array}{c}[A]^i\\ \vdots\\ B\end{array}}{A \to B}\,\to_{\mathcal{I},i} \qquad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ A\end{array}}{B}\,\to_{\mathcal{E}} \quad \Longrightarrow_{\beta} \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ A\\ \vdots\\ B\end{array}$$

$$\dfrac{\dfrac{[A]^i \qquad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ A \to B\end{array}}{B}\,\to_{\mathcal{E}}}{A \to B}\,\to_{\mathcal{I},i} \quad \Longrightarrow_{\eta} \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ A \to B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ B\end{array}}{A \wedge B}\,\wedge_{\mathcal{I}}}{A}\,\wedge_{\mathcal{E}} \quad \Longrightarrow_{\beta} \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ A\end{array}$$

Normalization Isomorphic to λ-Reduction

$$\dfrac{\dfrac{\begin{array}{c}[x:A]^i\\ \vdots\\ P(x):B\end{array}}{\lambda x P(x):A\to B}\to_{\mathcal{I},i} \qquad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ a:A\end{array}}{(\lambda x P(x))(a):B}\to_{\mathcal{E}} \quad \Longrightarrow_\beta \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ a:A\\ \vdots\\ P(a):B\end{array}$$

$$\dfrac{\dfrac{[x:A]^i \qquad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ P:A\to B\end{array}}{P(x):B}\to_{\mathcal{E}}}{\lambda x P(x):A\to B}\to_{\mathcal{I},i} \quad \Longrightarrow_\eta \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ P:A\to B\end{array}$$

Curry Howard: → and ∧

Proof Rules

$$\dfrac{\begin{array}{c}[x:A]^i\\ \vdots\\ P(x):B\end{array}}{\lambda x P(x):A\to B}\to_{\mathcal{I},i}$$

$$\dfrac{a:A \qquad P:A\to B}{P(a):B}\to_{\mathcal{E}}$$

$$\dfrac{P:A \qquad Q:B}{\langle P,Q\rangle:A\wedge B}\wedge_{\mathcal{I}}$$

$$\dfrac{P:A\wedge B}{\mathsf{fst}(P):A}\wedge_{\mathcal{E}} \qquad \dfrac{P:A\wedge B}{\mathsf{snd}(P):B}\wedge_{\mathcal{E}}$$

Normalization Rules

$$\dfrac{\dfrac{\begin{array}{c}[x:A]^i\\ \vdots\\ P(x):B\end{array}}{\lambda x P(x):A\to B}\to_{\mathcal{I},i} \qquad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ a:A\end{array}}{(\lambda x P(x))(a):B}\to_{\mathcal{E}} \quad \Longrightarrow_\beta \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ a:A\\ \vdots\\ P(a):B\end{array}$$

$$\dfrac{\dfrac{[x:A]^i \qquad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ P:A\to B\end{array}}{P(x):B}\to_{\mathcal{E}}}{\lambda x P(x):A\to B}\to_{\mathcal{I},i} \quad \Longrightarrow_\eta \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ P:A\to B\end{array}$$

$$\dfrac{\dfrac{\begin{array}{c}\mathcal{D}_1\\ \vdots\\ P:A\end{array} \qquad \begin{array}{c}\mathcal{D}_2\\ \vdots\\ Q:B\end{array}}{\langle P,Q\rangle:A\wedge B}\wedge_{\mathcal{I}}}{\mathsf{fst}(\langle P,Q\rangle):A}\wedge_{\mathcal{E}} \quad \Longrightarrow_\beta \quad \begin{array}{c}\mathcal{D}_1\\ \vdots\\ P:A\end{array}$$

Limitations of Curry Howard

Curry-Howard gives non-trivial identity criteria for proofs, but:

- It only works for certain logics
  e.g. intuitionistic logic, but not classical logic
- It only works for natural deduction proofs
- It handles 'parasitic' rules inelegantly

$$\dfrac{A\vee B \qquad \begin{array}{c}[A]^i\\ \vdots\\ C\end{array} \qquad \begin{array}{c}[B]^j\\ \vdots\\ C\end{array}}{C}\vee_{\mathcal{E}i,j}$$

C is a **parasitic** formula:

has nothing to do with the formula, $A\vee B$, being eliminated

Linear Logic and Proof Identity

Girard's motivation for linear logic:

More general identity criteria for proofs

Method:

Step back from natural deduction to sequent calculus
Remove structural rules to get linear logic
Show how this enforces interesting identity criteria
Other logics can be coded in linear logic

Can't cover all of this here.

Linear Logic: Overview

- Sequent calculus
- Removing structural rules
  Additive and multiplicative connectives
- Natural deduction for $\multimap$, $\otimes$ fragment

Sequent Calculus

Alternative proof system: sequential version of natural deduction

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\wedge_{\mathcal{L}} \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\wedge_{\mathcal{R}}$$

$$\frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\to_{\mathcal{L}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\to_{\mathcal{R}}$$

Rule $\to_{\mathcal{R}}$ discharges 'assumption' A

How to do Proofs in Sequent Calculus

$$\dfrac{\dfrac{\dfrac{A \vdash A}{A \wedge B \vdash A}\wedge_{\mathcal{L}} \qquad \dfrac{\dfrac{B \vdash B \qquad \dfrac{C \vdash C}{B, C \vdash C}\textit{Weakening}_{\mathcal{L}}}{B, B \to C \vdash C}\to_{\mathcal{L}}}{A \wedge B, B \to C \vdash C}\wedge_{\mathcal{L}}}{A \wedge B, A \to (B \to C) \vdash C}\to_{\mathcal{L}}}{A \to (B \to C) \vdash (A \wedge B) \to C}\to_{\mathcal{R}}$$

Work backwards from conclusion
Choose main connective to 'split' on

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\wedge_{\mathcal{L}} \qquad \frac{\Gamma \vdash A \qquad \Gamma, B \vdash C}{\Gamma, A \to B \vdash C}\to_{\mathcal{L}} \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \to B}\to_{\mathcal{R}}$$

Poor Identity Criteria from Sequent Calculus

Both the sequent derivations

$$\dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\wedge_{\mathcal{R}}}{A \wedge A', B \vdash A \wedge B}\wedge_{\mathcal{L}}}{A \wedge A', B \wedge B' \vdash A \wedge B}\wedge_{\mathcal{L}} \qquad \dfrac{\dfrac{\dfrac{A \vdash A \qquad B \vdash B}{A, B \vdash A \wedge B}\wedge_{\mathcal{R}}}{A, B \wedge B' \vdash A \wedge B}\wedge_{\mathcal{L}}}{A \wedge A', B \wedge B' \vdash A \wedge B}\wedge_{\mathcal{L}}$$

correspond to the same natural deduction proof

$$\dfrac{\dfrac{A \wedge A'}{A}\wedge_{\mathcal{E}} \qquad \dfrac{B \wedge B'}{B}\wedge_{\mathcal{E}}}{A \wedge B}\wedge_{\mathcal{I}}$$

Different sequentializations of one ND proof

Structural Rules

Structural Rules

$\Gamma \vdash \Delta$: *a subset of* Δ *follows from a subset of* Γ

Weakening: taking subsets
(allows fake dependencies)

$$\frac{\Gamma \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{Weakening}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\ \textit{Weakening}_{\mathcal{R}}$$

Contraction: repeating set elements
(allows duplication & multiple discharge of assumptions)

$$\frac{\Gamma, A, A \vdash \Delta}{\Gamma, A \vdash \Delta}\ \textit{Contraction}_{\mathcal{L}} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\ \textit{Contraction}_{\mathcal{R}}$$

Consequences of Removing Structural Rules

Structural rules implicit in traditional logic.

If we remove them

- Derivations are from *multisets* of premises,
  not sets of premises
- We preclude fake dependencies and multiple discharge of assumptions.
- We tighten identity criteria for proofs.
- We get two distinct versions of each familiar connective.
- And we get linear logic.

Contraction & Weakening: Premises as Resources

$$\phi_1, \ldots, \phi_n \vdash \psi$$

In traditional logic:

— ψ follows from some *subset* of $\{\phi_1, \ldots, \phi_n\}$
— sets allow permutation and duplication of premises (contraction)
— subsets allow discarding premises (weakening)

In linear logic

— ψ follows from the *multiset* $\{\phi_1, \ldots, \phi_n\}$
— multisets allow permutation of premises only

No duplication or discarding: premises become resources.

Permutation: order of premises remains immaterial

Alternate Rule for Conjunction ($\wedge_{\mathcal{R}}'$)

Usual
(Additive)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \wedge B}\ \wedge_{\mathcal{R}}$$

Alternate
(Multiplicative)

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \wedge B}\ \wedge_{\mathcal{R}}'$$

Where $\wedge_{\mathcal{R}}$ is a special case of $\wedge_{\mathcal{R}}'$: $\Delta = \Gamma$.

Rules $\wedge_{\mathcal{R}}$ and $\wedge_{\mathcal{R}}'$ interderivable given contraction and weakening:

Interderivability given Contraction & Weakening

- $\wedge_{\mathcal{R}}$ from $\wedge_{\mathcal{R}}'$

$$\dfrac{\dfrac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma, \Gamma \vdash A \wedge B}\ \wedge_{\mathcal{R}}'}{\Gamma \vdash A \wedge B}\ \textit{Contraction}_{\mathcal{L}}$$

- $\wedge_{\mathcal{R}}'$ from $\wedge_{\mathcal{R}}$

$$\dfrac{\dfrac{\Gamma \vdash A}{\Gamma, \Delta \vdash A}\ \textit{Weakening}_{\mathcal{L}} \qquad \dfrac{\Gamma \vdash B}{\Gamma, \Delta \vdash B}\ \textit{Weakening}_{\mathcal{L}}}{\Gamma, \Delta \vdash A \wedge B}\ \wedge_{\mathcal{R}}$$

Similarly for $\wedge_{\mathcal{L}}$

$$\frac{\Gamma, A \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\mathcal{L}} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \wedge B \vdash C}\ \wedge_{\mathcal{L}}'$$

Again, equivalent given contraction and weakening

Multiplicative and Additive Conjunction

Without contraction & weakening, rules define alternate connectives

Multiplicative conjunction (tensor)

$$\frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ \otimes_{\mathcal{R}} \qquad \frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\ \otimes_{\mathcal{L}}$$

Γ and Δ bundled to prove $A \otimes B$

Premises A and B bundled together

Additive conjunction (with)

$$\frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \& B}\ \&_{\mathcal{R}} \qquad \frac{\Gamma, A \vdash C}{\Gamma, A \& B \vdash C}\ \&_{\mathcal{L}}$$

Γ proves A and proves B
But not both at once

Γ plus something that can prove A
(and can prove B) proves C

A Plethora of Connectives

$A \multimap B$	Consume A to produce B
$A \otimes B$	Both A and B simultaneously available
$A \& B$	Either A or B available: your choice which
$A \oplus B$	Either A or B available: random selection
$!A$	Can duplicate or discard A

Menu: £5	$(\pounds \otimes \pounds \otimes \pounds \otimes \pounds \otimes \pounds)$
	$\multimap$
Fish	$[Fish$
	$\otimes$
Chips	$Chips$
	$\otimes$
Soup or Salad	$(Soup \,\&\, Salad)$
	$\otimes$
Fruit or cheese (depending on availability)	$(Fruit \oplus Cheese)$
	$\otimes$
Coffee (free refills)	$Coffee$ $\otimes !Coffee]$

Natural deduction for $\multimap$, $\otimes$ fragment

(Prawitz) Natural Deduction for (Multiplicative) Linear Logic

$$\dfrac{\begin{array}{c}[x:A]^i \\ \vdots \\ P:B\end{array}}{\lambda x.P : A \multimap B}\multimap_{\mathcal{I},i} \qquad \dfrac{a:A \qquad P:A \multimap B}{P(a):B}\multimap_{\mathcal{E}}$$

$$\dfrac{P:A \qquad Q:B}{P \times Q : A \otimes B}\otimes_{\mathcal{I}} \qquad \dfrac{P:A \otimes B \qquad \begin{array}{c}[x:A]^i[y:B]^j \\ \vdots \\ Q:C\end{array}}{\text{let } P \text{ be } x \times y \text{ in } Q : C}\otimes_{\mathcal{E},i,j}$$

- $\lambda x.f(..x..) : A \multimap B$
  $\lambda x.f(..x..)$ must be linear — exactly one occurrence of x bound
- $a \times b : A \otimes B$
  Pairing $a \times b$ without projections onto a and b
  Eliminated instead by: let $a \times b$ be $u \times v$ in $f \implies_\beta f[a/u, b/v]$

Linear and Non-Linear Assumptions

- Premises are just undischarged assumptions
- Needn't discharge assumptions in order they are introduced
- Non-Linear only:
  Can discharge multiple (co-indexed) assumptions at once

$$\dfrac{\dfrac{\dfrac{A \to (B \to C) \qquad \dfrac{[A \wedge B]^1}{A}\wedge_{\mathcal{E}}}{B \to C}\to_{\mathcal{E}} \qquad \dfrac{[A \wedge B]^1}{B}\wedge_{\mathcal{E}}}{C}\to_{\mathcal{E}}}{(A \wedge B) \to C}\to_{\mathcal{I},1}$$

- Non-Linear only:
  Can discharge non-existent assumptions

$$\dfrac{\dfrac{[A]^1}{B \to A}\to_{\mathcal{I},2}}{A \to (B \to A)}\to_{\mathcal{I},1}$$

What You Need to Remember

Linear Logic

Traditional:	Linear:
$A, A \to B \vdash B$	$A, A \multimap B \vdash B$
$A, A \to B \vdash A \wedge B$	$A, A \multimap B \nvdash A \otimes B$
Re-use A	Cannot re-use A
$A, B \vdash A$	$A, B \nvdash A$
Discard B	Cannot discard B

$$A \multimap (B \multimap C) \equiv$$
$$B \multimap (A \multimap C) \equiv$$
$$(A \otimes B) \multimap C$$

Curry Howard

$$\frac{a : A \qquad P : A \multimap B}{P(a) : B} \multimap_{\mathcal{E}}$$

$$\frac{\begin{array}{c}[x : A]^i \\ \vdots \\ P(x) : B\end{array}}{\lambda x P(x) : A \multimap B} \multimap_{\mathcal{I},i}$$

Exercise

Using normalization or λ-reduction, show the essential equivalence of $A \multimap (B \multimap C) \equiv B \multimap (A \multimap C) \equiv (A \otimes B) \multimap C$