1.

Exact linearization: - in the region

$$\{ x \in \mathbb{R}^4 : |x_1 x_4| > \delta \}$$

define a coordinate system:

$$\Phi : x \to (\xi, \eta) \text{ such that}$$

$$\xi_1 = x_1 - r_0$$
$$\xi_2 = x_2$$
$$\xi_3 = B x_1 x_4^2 - BG \sin x_3$$
$$\eta = x_4$$

- check that Φ is full rank away from $x_3 = \frac{\pi}{2}$ and $x_3 = -\frac{\pi}{2}$

$\therefore$ system dynamics in mixed coordinates are:

$$\dot{\xi}_1 = \xi_2$$
$$\dot{\xi}_2 = \xi_3$$
$$\dot{\xi}_3 = B x_2 x_4^2 - BG x_4 \cos x_3 + 2B x_1 x_4 u.$$
$$\dot{\eta} = u$$

let $u = \frac{1}{2B x_1 x_4} \left(-B x_2 x_4^2 + BG x_4 \cos x_3 + v \right)$

$\therefore$ zero dynamics are 1-D:

$$\dot{x}_4 = \frac{G}{2r_0} \cos x_3 \quad \text{where} \quad r_0 x_4^2 = G \sin x_3$$

- problem can't track anything in which $x_4 \to 0$

2.

$$\ddot{y} + 2\xi\dot{y} + (1-y)y = 0$$

$$0 < \xi < 1$$

$$x_1 \triangleq y$$

$$x_2 \triangleq \frac{\dot{y} + \xi y}{\sqrt{1-\xi^2}}$$

$$\Rightarrow \quad \dot{x}_1 = -\xi x_1 + \sqrt{1-\xi^2}\, x_2$$

$$\dot{x}_2 = -\sqrt{1-\xi^2}\, x_1 - \xi x_2 + \frac{x_1^2}{\sqrt{1-\xi^2}}$$

let $\gamma \triangleq \sqrt{1-\xi^2}$

$$\Rightarrow \quad \dot{x}_1 = -\xi x_1 + \gamma x_2$$

$$\dot{x}_2 = -\gamma x_1 - \xi x_2 + \frac{x_1^2}{\gamma}.$$

eq. at $(0,0)$, $\left(1, \frac{\xi}{\gamma}\right)$

stable focus $\quad$ saddle.

$$-\gamma - \frac{\xi^2}{\gamma} + \frac{1}{\gamma}$$

$$\frac{-\gamma^2 - \xi^2 + 1}{\gamma}$$

$$= \cancel{\frac{-(1-\xi^2) - \xi^2 + 1}{\sqrt{1-\xi^2}}} \quad 0$$

$$Df\Big|_{(0,0)} \triangleq A = \begin{bmatrix} -\xi & \gamma \\ -\gamma & -\xi \end{bmatrix}$$

choose $Q = I$, then

lyap. eqn. $PA + A^T P = -I$

solve. $P = \frac{1}{2\xi} I$ (by writing it out element-wise).

Thus. $$V(x) = x^T P x = \frac{x_1^2 + x_2^2}{2\xi}.$$

$$\therefore \quad \dot{V}(x) = -(x_1^2 + x_2^2) + \frac{x_1^2 x_2}{\xi\gamma}$$

Set $\dot{V} = 0$; solve for x_1, subst. into $V(x)$, we get

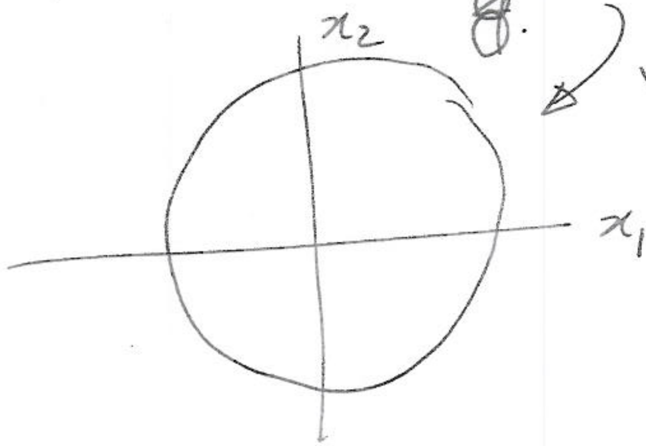
$$V(x_2) = \frac{x_2^3}{2\xi(x_2 - \xi\gamma)} \quad (\text{when } \dot{V} = 0)$$

$$(\text{for } x_2 > \xi\gamma)$$

$$\min_{x_2} V(x_2) \Rightarrow x_2 = \frac{3\xi\gamma}{2} \Rightarrow V = \frac{27\xi\gamma^2}{8}$$

$$\therefore \quad x_1^2 + x_2^2 < \frac{27\xi\gamma^2}{8}.$$

an estimate of the domain of attraction



(b) Using the Lyapunov function

$$V = y^2 - \frac{2}{3} y^3 + \dot{y}^2$$

we have

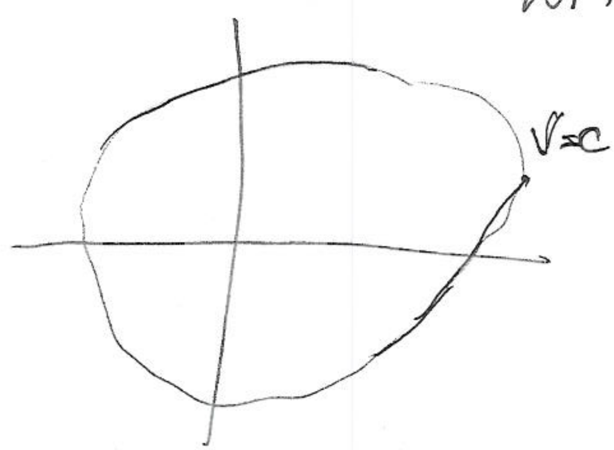
$$\dot{V} = -4\xi \dot{y}^2 \leq 0$$

The contours of V are given by

$$\dot{y}^2 = \frac{2}{3} y^3 - y^2 + C.$$

$$\dot{y}^2 = \frac{(y-1)^2 (2y+1)}{3}$$

largest closed one obtained with $C = \frac{1}{3}$?



E209A Final Solutions TOMLIN

$$y_m^{(\gamma)} = c_m A_m^{\gamma} x_m + c_m A_m^{\gamma-1} b_m r + c_m A_m^{\gamma-2} b_m \dot{r} + \cdots + c_m A_m b_m r^{(\gamma-2)} + c_m b_m r^{(\gamma-1)}$$

Thus, to get $y(t)$ to track $y_m(t)$, we need $y^{(\gamma)} = v$ where

$$v = y_m^{(\gamma)} + A_{\gamma-1}\left(y_m^{(\gamma-1)} - y^{(\gamma-1)}\right) + A_{\gamma-2}\left(y_m^{(\gamma-2)} - y^{(\gamma-2)}\right) + \cdots + A_1(y_m - y)$$

To prevent the appearance of $\dot{r}, \ddot{r} \ldots$ in the control law we need that.

$$c_m A_m^{\gamma-2} b_m = 0$$
$$\vdots$$
$$c_m A_m b_m = 0$$
$$c_m b_m = 0$$

(or that the linear system has relative degree $\geq \gamma$)

Thus,

$$u = \frac{1}{L_g L_f^{\gamma-1} h(x)} \left[y_m^{(\gamma)} - L_f^{\gamma} h(x) + v \right]$$

contains terms relative to SISO linear reference model

from the SISO NL system

1. TRY $V(w) = \frac{1}{2}(J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2)$
(a) LPDF ✓ as a lyapunov function candidate

$$\dot{V}(w) = J_1 w_1 \dot{w}_1 + J_2 w_2 \dot{w}_2 + J_3 w_3 \dot{w}_3$$
$$= 0$$

(i) $\Rightarrow$ origin is stable

it is not asymptotically stable — SHOW!

(ii) $u_i = -k_i w_i$

Using same $V(w)$, we have

$$\dot{V} = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 < 0$$

$-\dot{V}$ is PD

$\therefore$ origin is globally asymptotically stable

(b) (i) $\dot{x}_1 = x_2, \quad \dot{x}_2 = -g(x_1) - h(x_1) x_2$

@ equilibrium,

$$x_2 = 0 \;\; \& \;\; g(x_1) + h(x_1) x_2 = 0$$
$$\Rightarrow x_2 = 0 \;\; \& \;\; g(x_1) = 0$$

if we assume that $g(x_1) = 0$ has an isolated root at the origin, then the origin is an isolated equilibrium point.

(ii) Take candidate lyapunov fn as, in class,

$$V(x) = \int_0^{x_1} g(y)\,dy + \frac{1}{2} x_2^2$$

$$\therefore \dot{V}(x) = g(x_1) x_2 - x_2 g(x_1) - h(x_1) x_2^2$$
$$= -h(x_1) x_2^2 \leq 0$$

now, $\dot{V}(x) = f^T(x)\left\{ P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \right\} f(x)$

$$\leq -\|f(x)\|_2^2$$

Since $f(x) = 0 \Leftrightarrow x = 0$, $\dot{V}(x) < 0 \quad \forall x \in \mathbb{R}^n$, $x \neq 0$

$\therefore$ The origin is globally asymptotically stable

3. (a) $3x_1 = x_2$, $x_1 - x_1^3 + x_2 = 0$

$\Rightarrow x_1(4 - x_1^2) = 0$

eq pts. $\Rightarrow (0,0), (2,6), (-2,-6)$.

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 1-3x_1^2 & 1 \\ 3 & -1 \end{bmatrix}$$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \Rightarrow \lambda = \pm 2 \Rightarrow$ saddle

$\left.\frac{\partial f}{\partial x}\right|_{(2,6)} = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix} \Rightarrow \lambda = -11.29, -0.71 \Rightarrow$ stable node

$\left.\frac{\partial f}{\partial x}\right|_{(-2,-6)} = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix} \Rightarrow$ same as above stable node

(b). Let $A = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}$ & P solves

$$PA + A^T P = -I$$

... solve (I used MATLAB) to get:

$$P = \begin{bmatrix} 0.0938 & 0.1771 \\ 0.1771 & 0.6771 \end{bmatrix}$$

$\lambda_{max}(P) = 0.7266$

$\lambda_{min}(P) = 0.0442$

1b cont^{d}

now, $\boxed{\text{if } h(x_1) > 0}$ $\forall x_1 \in D$ (D contains origin)

Then $\dot{V} \le 0$ and

$$\dot{V} \equiv 0 \Rightarrow h(x_1) x_2^2 \equiv 0 \Rightarrow x_2 \equiv 0$$
$$\Rightarrow g(x_1) = 0$$
$$\Rightarrow x_1 = 0$$

hence, by Lasalle, origin is asymp. stable.

2. $\dot{x} = f(x)$ with $f(0) = 0$

(a) $x^T P f(x) + f^T(x) P x$

$$= x^T P \int_0^1 \frac{\partial f}{\partial x}(\sigma x)\, x \, d\sigma + \int_0^1 x^T \left[\frac{\partial f}{\partial x}(\sigma x)\right]^T d\sigma \, P x.$$

$$= x^T \int_0^1 \left\{ P \frac{\partial f}{\partial x}(\sigma x) + \left[\frac{\partial f}{\partial x}(\sigma x)\right]^T P \right\} d\sigma \, x$$

$$\le -x^T x \qquad \forall x \in \mathbb{R}^n$$

(b) now, $V(x) = f^T(x) P f(x)$ is positive semi-definite

To show it is positive definite, need to show $V = 0 \Leftrightarrow x = 0$

since $P > 0$, we need to show $f(x) = 0 \Leftrightarrow x = 0$, that is 0 is the unique equilibrium.

By contradiction:

suppose $\exists p \neq 0$ st $f(p) = 0$

Then $p^T p \le -[p^T P f(p) + f^T(p) P p] = 0$

$\Rightarrow p = 0$

$\Rightarrow$ contradiction.

$\therefore V = 0 \Leftrightarrow x = 0 \therefore V$ is PD

3(b) cont^{d}.

for (2,6) → shift to origin:

$$\left.\begin{array}{l}\bar{x}_1 = x_1 - 2 \\ \bar{x}_2 = x_2 - 6\end{array}\right\} \Rightarrow \begin{array}{l}\dot{\bar{x}}_1 = -11\bar{x}_1 + \bar{x}_2 - 6\bar{x}_1^2 - \bar{x}_1^3 \\ \dot{\bar{x}}_2 = 3\bar{x}_1 - \bar{x}_2\end{array}$$

use $V = \bar{x}^T P \bar{x}$ as a L-fn candidate. PD ✓

$$\dot{V} = -\bar{x}^T\bar{x} - 2(p_{11}\bar{x}_1 + p_{12}\bar{x}_2)(6+\bar{x}_1)\bar{x}_1^2$$

$$\le -\|\bar{x}\|_2^2 - 12(p_{11}\bar{x}_1 + p_{12}\bar{x}_2)\bar{x}_1^2 - 2p_{12}\bar{x}_1^3\bar{x}_2$$

$$\le -\|\bar{x}\|_2^2 + 12\sqrt{p_{11}^2 + p_{12}^2}\,\|\bar{x}\|_2^3 + p_{12}\|\bar{x}\|_2^4$$

$$\le -(1 - 2.4r - 0.1771r^2)\|\bar{x}\|_2^2$$

for $\|\bar{x}\|_2 \le r$.

now, taking $r = 0.4$, we see that

$\dot{V}(\bar{x})$ is negative inside $\{\|\bar{x}\|_2 \le r\}$

if we choose $c < \lambda_{min}(P)r^2$ $(= 0.00707)$, then

$\{V(\bar{x}) \le c\} \subset \{\|\bar{x}\|_2 \le r\}$ because

$$\lambda_{min}(P)\|\bar{x}\|_2^2 \le V(\bar{x})$$

take $c = 0.007$, then, the region of attraction is estimated by $\{\bar{x}^T P \bar{x} \le 0.007\}$ (similarly for $(-2,-6)$)

(c) A less conservative estimate of the R.O.A can be obtained graphically:

i) plot the contour of $\dot{V}(\bar{x}) = 0$ in the x_1-x_2 plane

ii) choose c, and plot contour $V(\bar{x}) = c$, with increasing c until we obtain the largest c for which →

3(c) cont$^{\underline{d}}$

The surface $V(\bar{x})=C$ is inside the region $\{\dot{V}(\bar{x})=0\}$. $\boxed{C=0.1 \text{ is a value that satisfies this}}$

(plot phase portrait)

4. $$A(x)\Big|_{x=0} = \frac{\partial f}{\partial x}\Big|_{x_1=x_2=0} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$\Rightarrow \lambda = \pm i$

$\rightarrow$ linearization inconclusive

Choose $v(x) = \|x\|^2$ PDF.

and $\dot{v}(x) = 2\varepsilon(x_1^2+x_2^2)^2 \sin(x_1^2+x_2^2)$

now consider $\varepsilon \in [-1,1]$

I. if $\varepsilon=0$, $\dot{v}(x)=0 \rightarrow (0,0)$ is SISL

II if $\varepsilon \in [-1,0)$

$-\dot{v}(x) = 2|\varepsilon|\,\|x\|^4 \sin(\|x\|^2)$

now if $\|x\|$ is small, then $\sin\|x\|^2 \approx \|x\|^2$

$\Rightarrow -\dot{v}(x) \approx 2|\varepsilon|\,\|x\|^6$

$\Rightarrow -\dot{v}$ is LPDF

$\Rightarrow$ local asymp. stability

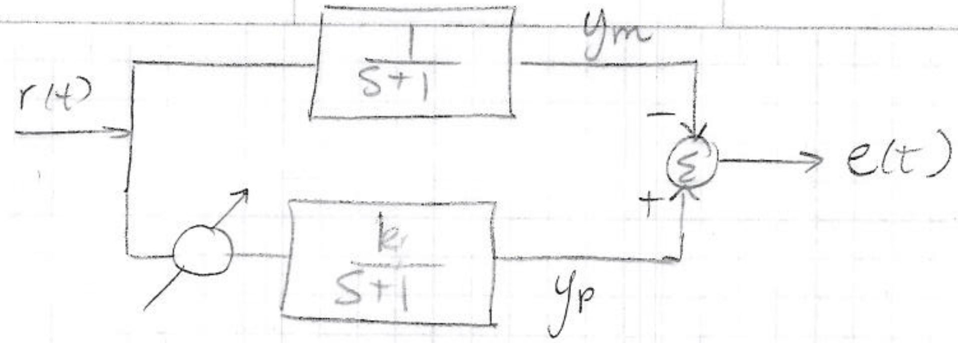
III $\varepsilon>0$ ie $\varepsilon \in (0,1]$

$\dot{v}(x) = 2\varepsilon\|x\|^4 \sin(\|x\|^2)$ is LPDF

by the Lyapunov Instability Thm

the origin is unstable

4.



let $\phi(t) := \theta(t) - \theta^*$ be the parameter error

(a) $y_m(s) = \frac{1}{1+s} r(s)$

$\therefore \dot{y}_m + y_m = r$

Similarly $y_p = \frac{k}{s+1} \theta(s) * r(s)$

$\therefore \dot{y}_p + y_p = k_p \theta(t) r(t)$

$(\dot{y}_p - \dot{y}_m) + (y_p - y_m) = (k \theta - 1) r$

$= k (\theta - \frac{1}{k}) r$

$\therefore \dot{e} + e = k \phi r \leftarrow$

(b) $\boxed{\text{Choose } \dot{\phi}(t) = \dot{\theta}(t) = -r(t) e(t)}$

$V = \frac{1}{2} e^2 + \frac{1}{2} k \phi^2$

if $k > 0$ then $V(x)$ is a PDF.

so as to make $\dot{V} = -e^2$

$\dot{V} = e\dot{e} + k \phi \dot{\phi}$

$= e(-e + k\phi r) + k\phi(-re)$

$= -e^2 \leq 0$

Now $\dot{V} \leq 0$. How to prove that $e \to 0$??

This is the HARD PART:

- you cannot use LASALLE because the system may not be time-invariant or periodic if $r(t)$ is not periodic.

So we have to do something more:

Since $V = \frac{1}{2}e^2 + \frac{1}{2}k\phi^2$ and $\dot{V} \leq 0$, $e(t), \phi(t)$ are bounded

Assume $r(t)$ is bounded, then y_m is bounded implying that

$y_p = y_m + e$ is bounded

Now use Barbalat's lemma and the generalization of Lasalle's principle (Thm 5.27) to show $e \to 0$ as $t \to \infty$.

↑ notes to E209A students (W'07)
you will not be expected to
know Barbarat's lemma